Univariate and Multivariate Analysis of Time Series and Complex Networks

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Outline

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1. Dependence measures in univariate time series

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2. Interdependence in multivariate time series

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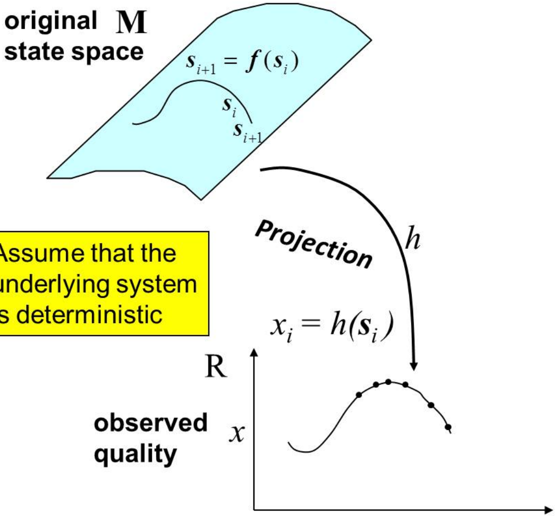
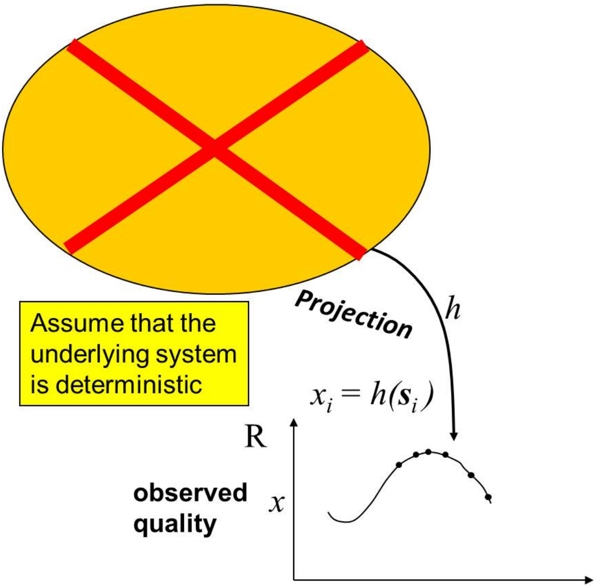
3. Complex networks from multivariate time series
4. High-dimensional time series: Implications and solutions

State space reconstruction (embedding)



Assume that the underlying system is deterministic

State space reconstruction (embedding)



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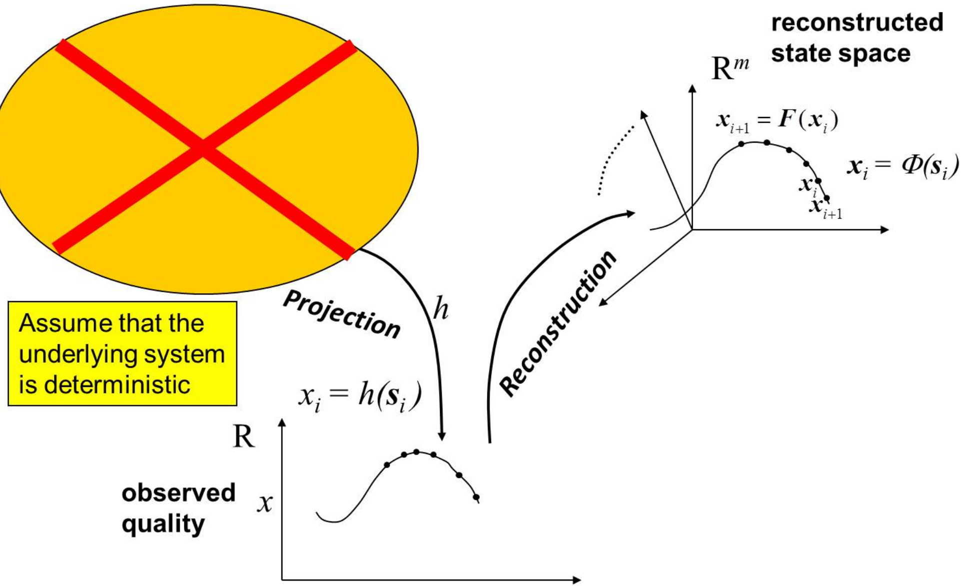

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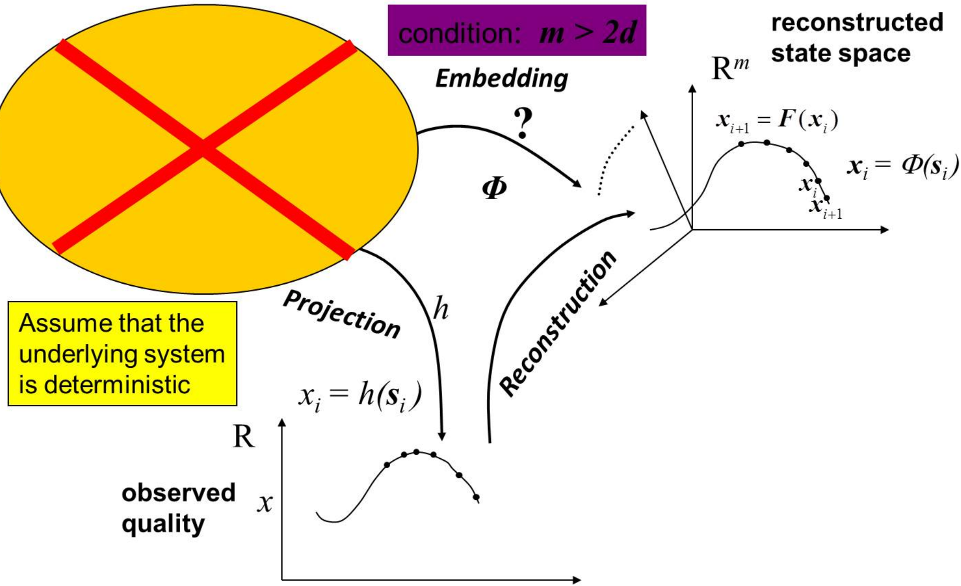

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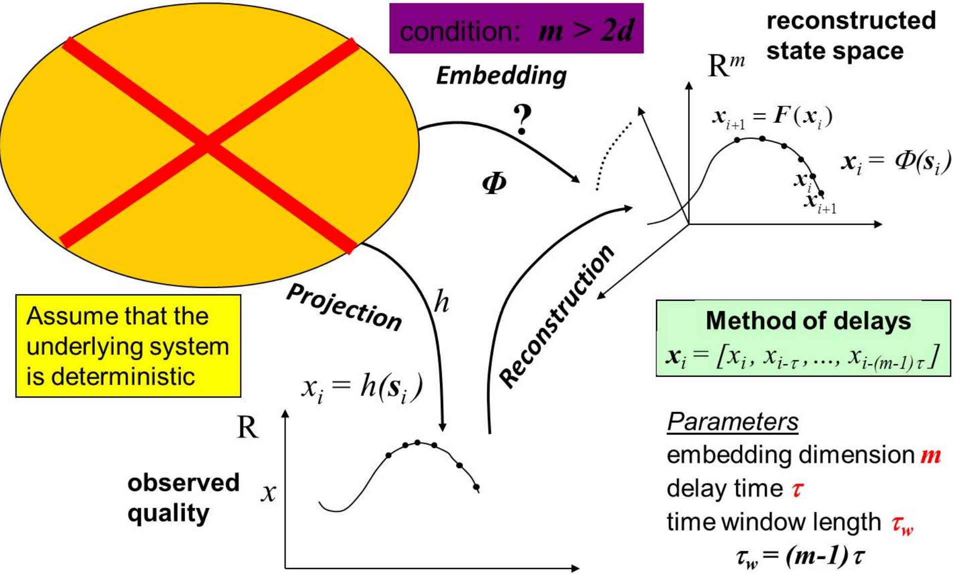
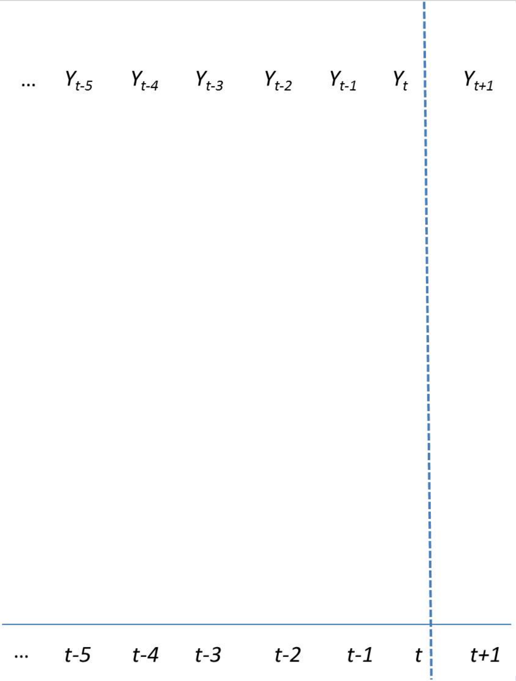




… Y_{t-5} Y_{t-4} Y_{t-3} Y_{t-2} Y_{t-1} Y_t Y_{t+1}
… t-5 t-4 t-3 t-2 t-1 t t+1

Dynamical system

$\boldsymbol{s}_{t+1} = \boldsymbol{f}(\boldsymbol{s}_t), \quad \boldsymbol{s}_t \in \mathrm{R}^d$

$\dot{\boldsymbol{s}} = \boldsymbol{f}(\boldsymbol{s}), \quad \boldsymbol{s}_t \in \mathrm{R}^d$

Observable (assuming $y_t = h(\boldsymbol{s}_t)$)

$y_{t+1} = F(\boldsymbol{y}_t) = \mathrm{F}(y_t, y_{t-1}, \ldots, y_{t-m+1}) \quad y_t \in \mathrm{R}$

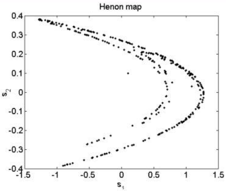
... Y_{t-5} Y_{t-4} Y_{t-3} Y_{t-2} Y_{t-1} Y_t Y_{t+1}

... $t-5$ $t-4$ $t-3$ $t-2$ $t-1$ t $t+1$

$$s(t) = 1 - 1.4\, s(t-1)^2 + 0.3 s(t-2)$$

or

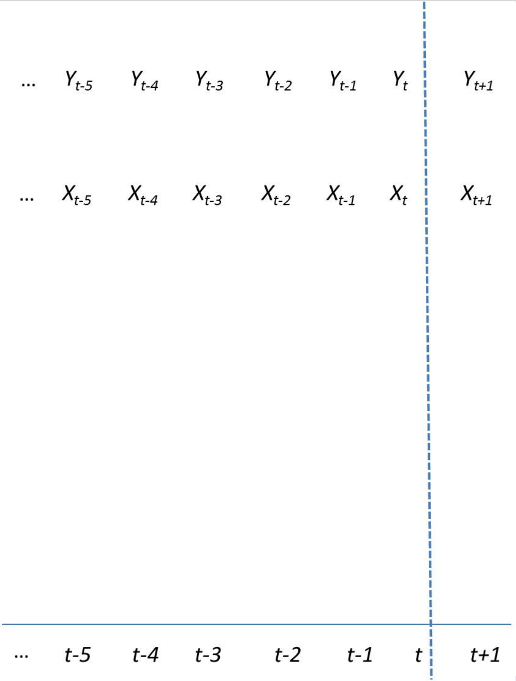
$$s_1(t) = 1 - 1.4\, s_1(t-1)^2 + s_2(t-1)$$
$$s_2(t) = 0.3\, s_1(t-1)$$



$y_t = s_t$

$\ldots \quad Y_{t-5} \quad Y_{t-4} \quad Y_{t-3} \quad Y_{t-2} \quad Y_{t-1} \quad Y_t \quad Y_{t+1}$

$\ldots \quad t-5 \quad t-4 \quad t-3 \quad t-2 \quad t-1 \quad t \quad t+1$



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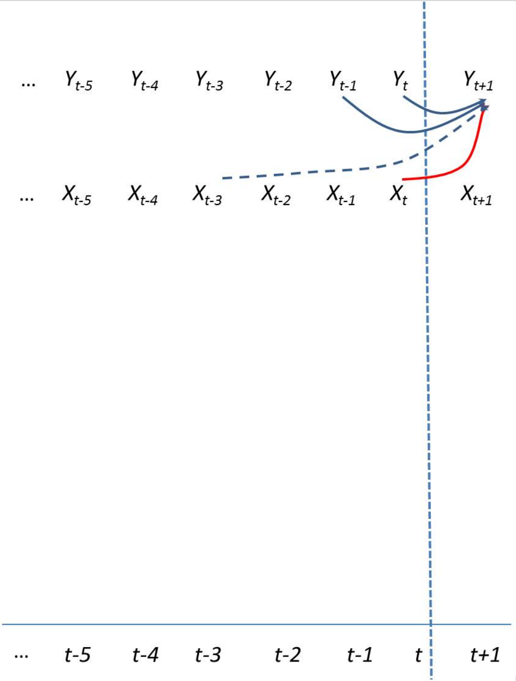
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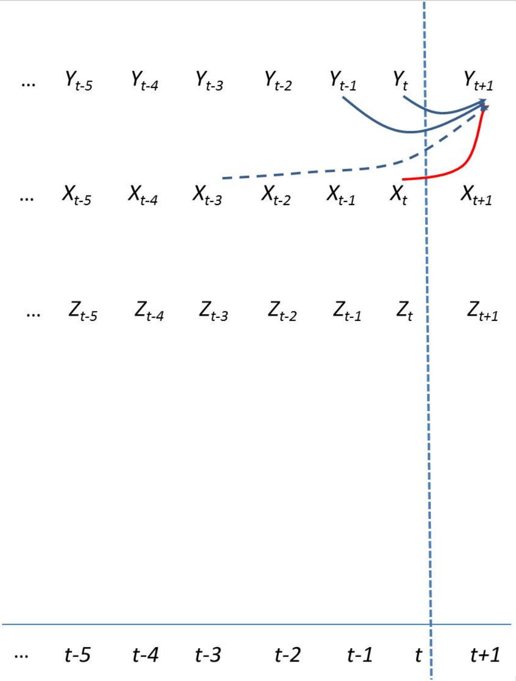

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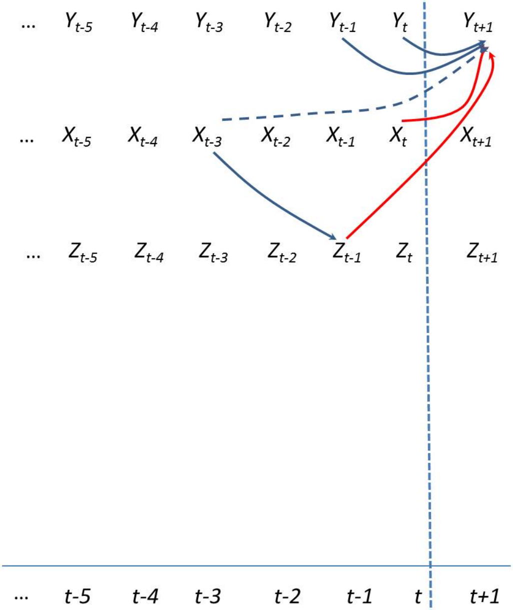

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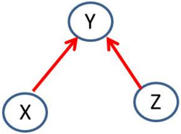
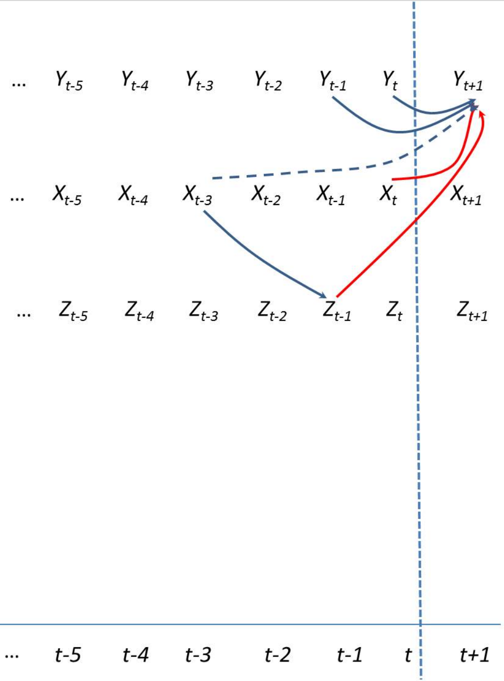

Y
X
Z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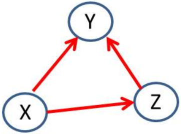
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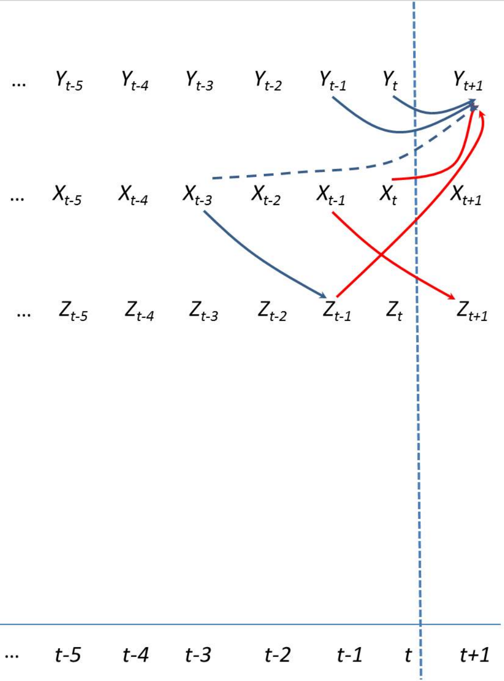
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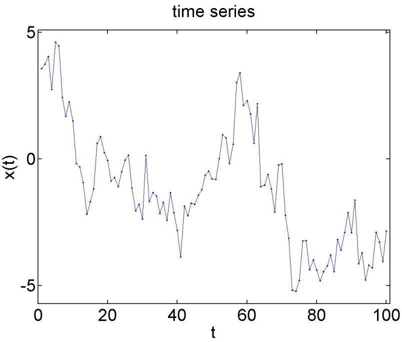
1. Dependence measures in univariate time series
2. Interdependence in multivariate time series
3. Complex networks from multivariate time series
4. High-dimensional time series: Implications and solutions

Are X_t and X_{t-1} linearly correlated?

Are X_t and X_{t-2} linearly correlated?

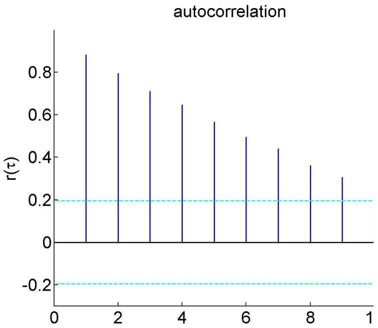


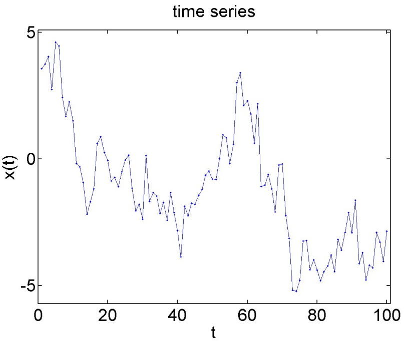
Autocorrelation $r(\tau) = r(X_t; X_{t-\tau})$

Are X_t and X_{t-1} linearly correlated? $r(1) = r(X_t; X_{t-1}) \neq 0$? Yes

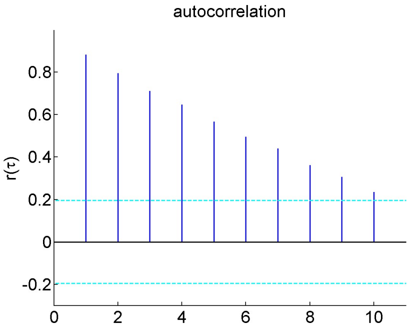
Are X_t and X_{t-2} linearly correlated? $r(2) = r(X_t; X_{t-2}) \neq 0$? Yes

$$r(\tau) = r(X_t; X_{t-\tau}) = \frac{1}{n-\tau} \sum_{t=\tau+1}^{n} (x_t - \bar{x})(x_{t-\tau} - \bar{x})/s_X^2$$





Are X_t and X_{t-2} directly linearly correlated?

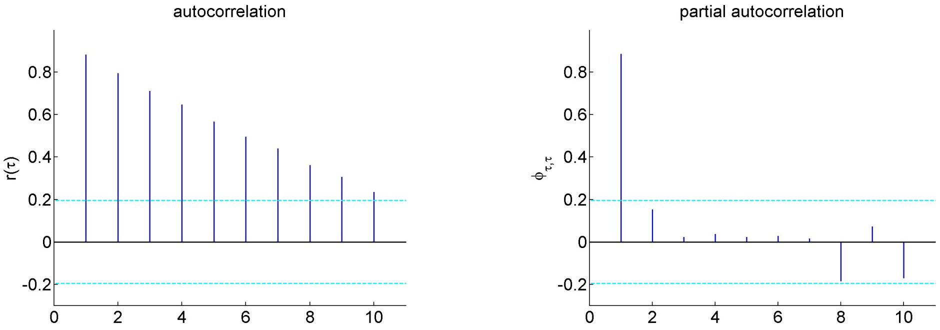


Partial autocorrelation $\phi_{\tau,\tau} = r(X_t; X_{t-\tau} | X_{t-1}, \ldots, X_{t-\tau+1})$

Are X_t and X_{t-2} directly linearly correlated?

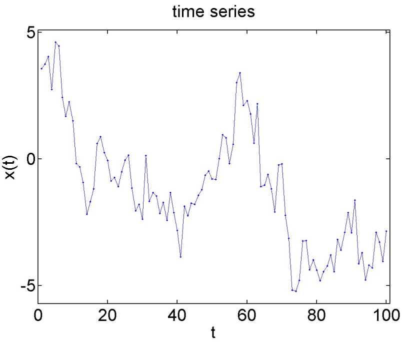
Are X_t and X_{t-2} linearly correlated given X_{t-1}?

$$r(X_t; X_{t-2} | X_{t-1}) \neq 0? \qquad \text{No}$$



Are X_t and X_{t-1} linearly and nonlinearly correlated?

Are X_t and X_{t-2} linearly and nonlinearly correlated?



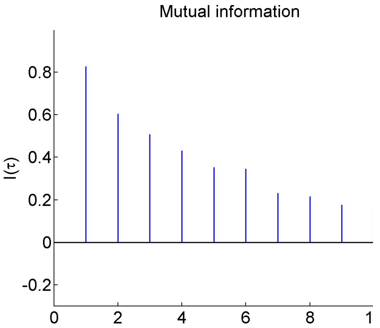
Mutual information $I(\tau) = I(X_t; X_{t-\tau})$

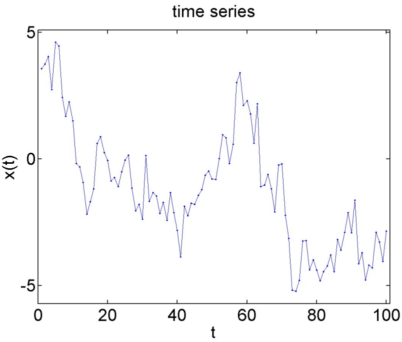
Are X_t and X_{t-1} linearly and nonlinearly correlated? $I(X_t; X_{t-1}) \neq 0$? Yes

Are X_t and X_{t-2} linearly and nonlinearly correlated? $I(X_t; X_{t-2}) \neq 0$? Yes

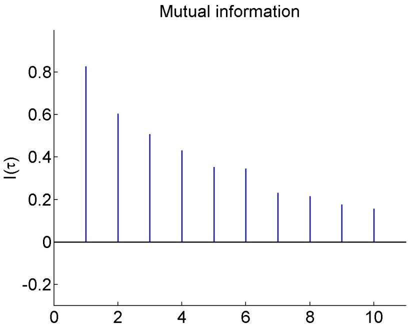
$$I(\tau) = I(X_t, X_{t-\tau}) = H(X_t) + H(X_{t-\tau}) - H(X_t, X_{t-\tau})$$

$$= \sum_{x_t, x_{t-\tau}} p_{X_t X_{t-\tau}}(x_t, x_{t-\tau}) \log \frac{p_{X_t X_{t-\tau}}(x_t, x_{t-\tau})}{p_{X_t}(x_t) p_{X_{t-\tau}}(x_{t-\tau})}$$





Are X_t and X_{t-2} directly linearly and nonlinearly correlated?

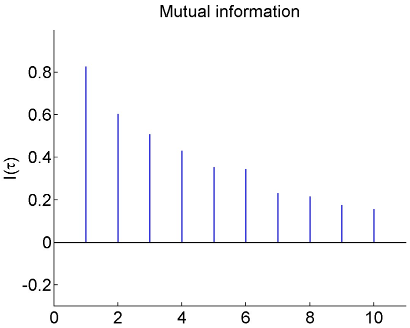


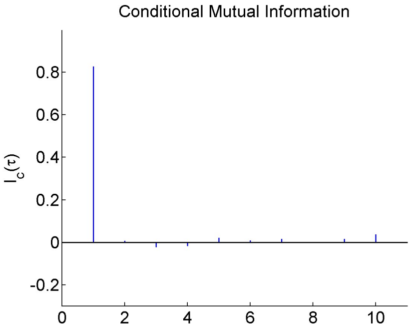
Conditional mutual information $I_c(\tau) = I(X_t; X_{t-\tau} | X_{t-1}, \ldots, X_{t-\tau+1})$

Are X_t and X_{t-2} directly linearly and nonlinearly correlated?

Are X_t and X_{t-2} linearly and nonlinearly correlated given X_{t-1}?

$$I(X_t; X_{t-2} | X_{t-1}) \neq 0? \qquad \text{No}$$



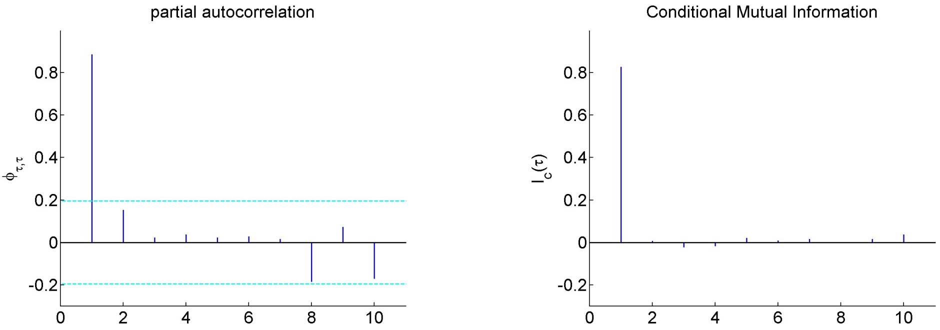


Are X_t and X_{t-2} directly linearly correlated?

$$r(X_t; X_{t-2}|X_{t-1}) \neq 0? \qquad \text{No}$$

Are X_t and X_{t-2} directly linearly or/and nonlinearly correlated?

$$I(X_t; X_{t-2}|X_{t-1}) \neq 0? \qquad \text{No}$$



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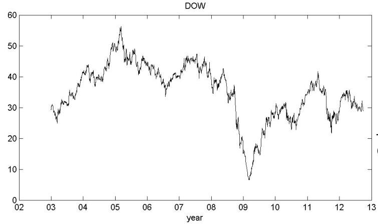
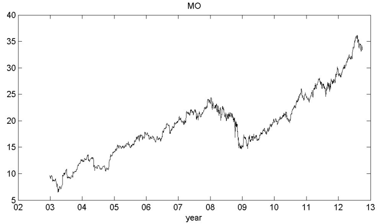
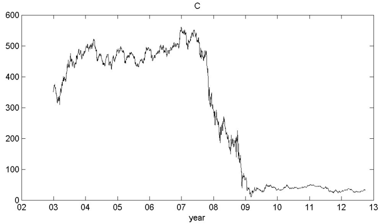
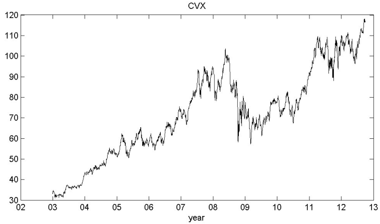


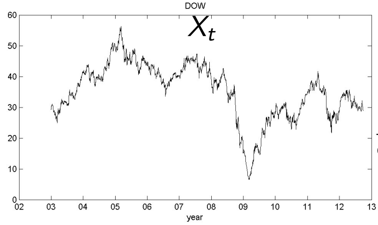
correlation
?



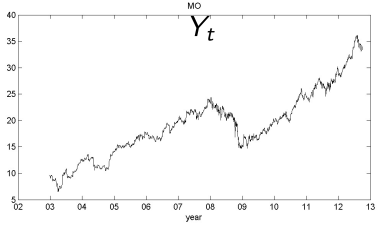


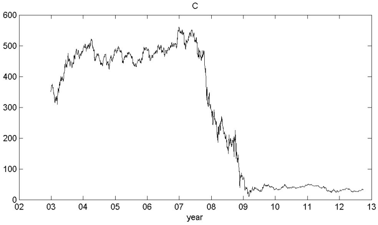


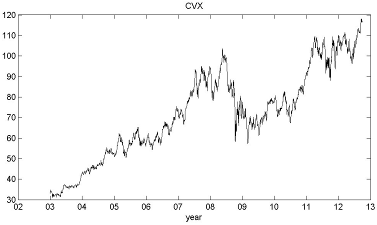
$r(X_t; Y_t)$



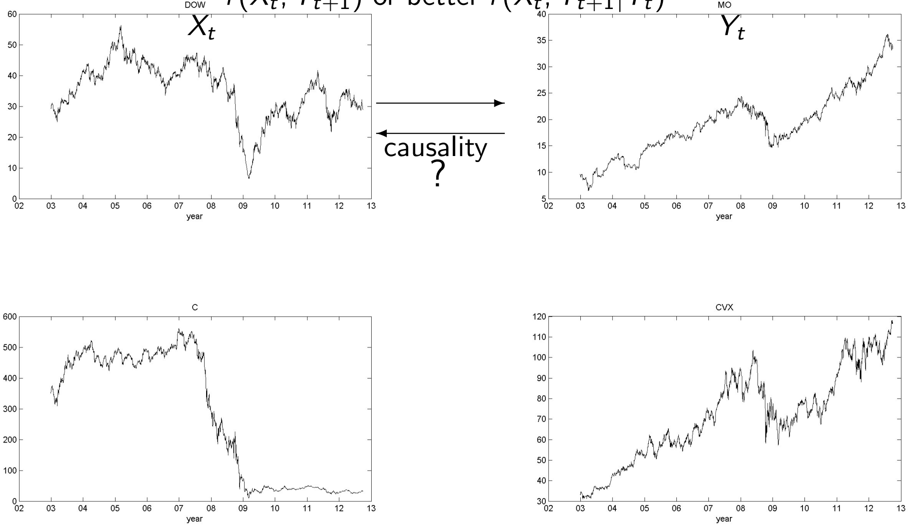
correlation
?



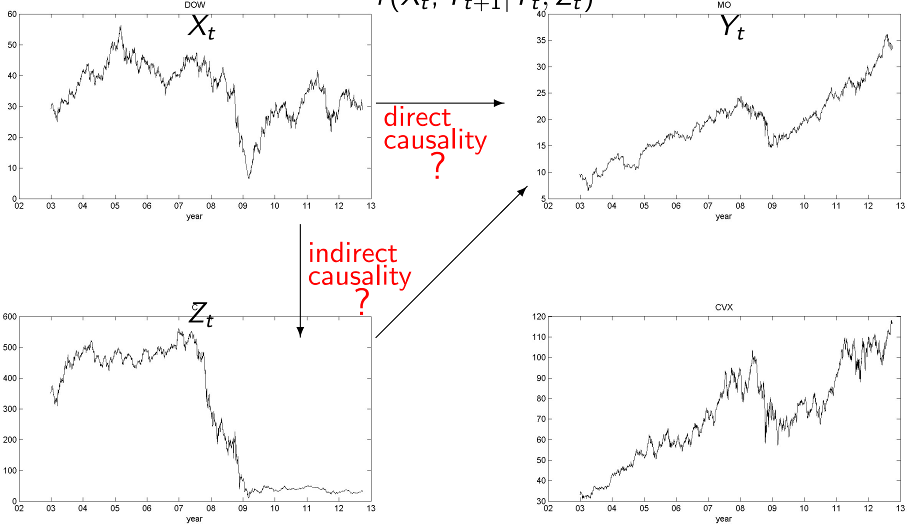




$r(X_t; Y_{t+1})$ or better $r(X_t; Y_{t+1}|Y_t)$



$$r(X_t; Y_{t+1} | Y_t, Z_t)$$


DOW
X_t
MO
Y_t
direct causality ?
indirect causality ?
C
Z_t
CVX
year

Correlation measures

Bivariate time series $\{x_t, y_t\}_{t=1}^{n}$

Linear correlation measures:

Estimate of cross-covariance

$$c_{XY}(\tau) = \hat{\gamma}_{XY}(\tau) = \frac{1}{n-\tau}\sum_{t=1}^{n-\tau}(x_t - \bar{x})(y_{t+\tau} - \bar{y})$$

$\bar{x}$ and $\bar{y}$ are sample means.

Estimate of **cross-correlation**:

$$r_{XY}(\tau) = \hat{\rho}_{XY}(\tau) = \frac{c_{XY}(\tau)}{c_{XY}(0)} = \frac{c_{XY}(\tau)}{s_X s_Y}$$

s_X and s_Y are sample standard deviations.

- $|r_{XY}(\tau)| \leq 1$
- $r_{XY}(\tau) = r_{YX}(-\tau)$ but $r_{XY}(\tau) \neq r_{XY}(-\tau)$

Nonlinear correlation measures:

Entropy: information from each sample of X (assume proper discretization of X)

$$H(X) = -\sum_{x} p_X(x) \log p_X(x)$$

Mutual information: information for Y knowing X and vice versa

$$I(X,Y) = H(X)+H(Y)-H(X,Y) = \sum_{x,y} p_{XY}(x,y) \log \frac{p_{XY}(x,y)}{p_X(x)p_Y(y)}$$

For $X \rightarrow X_t$ and $Y \rightarrow Y_{t+\tau}$,
cross-delayed mutual information:

$$I_{XY}(\tau) = I(X_t, Y_{t+\tau}) = \sum_{x_t, y_{t+\tau}} p_{X_t Y_{t+\tau}}(x_t, y_{t+\tau}) \log \frac{p_{X_t Y_{t+\tau}}(x_t, y_{t+\tau})}{p_{X_t}(x_t) p_{Y_{t+\tau}}(y_{t+\tau})}$$

To compute $I_{XY}(\tau)$ make a partition of $\{x_t\}_{t=1}^n$, a partition of $\{y_t\}_{t=1}^n$ and compute probabilities for each cell from the relative frequency.

$r_{XY}(0) \neq 0$:

$\Longrightarrow$ (linear) correlation of x_t and y_t

$\Longrightarrow$ systems X and Y are correlated, $X \sim Y$

$r_{XY}(\tau) \neq 0$:

$\Longrightarrow$ (linear) correlation of x_t and $y_{t+\tau}$

$\Longrightarrow$ X effects the future of Y

$\Longrightarrow$ $X \rightarrow Y$

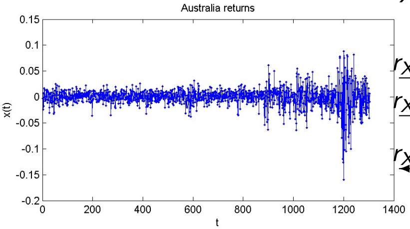
$$r_{XY}(-\tau) \neq 0 \quad \Longrightarrow \quad Y \rightarrow X$$

Thus $r_{XY}(\tau)$ and $I_{XY}(\tau)$ indicate the direction of interaction.

Can they also be used as causality measures?
Not the most appropriate, but they have been used in many studies

Example: Returns for USA, UnitedKingdom, Greece and Australia.

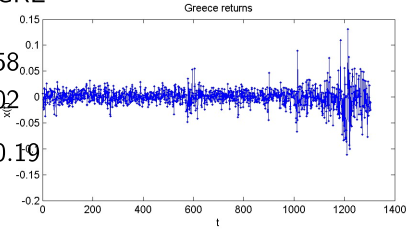
X:AUS, Y:GRE



$r_{XY}(0) = 0.58$

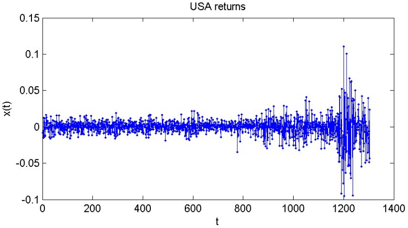
$r_{XY}(1) = 0.02$

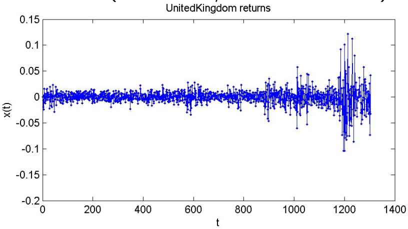
$r_{XY}(-1) = 0.19$



returns:
$x_t = \log(y_t) - \log(y_{t-1})$

Is the measure significant?
Can I draw a link? (directed/non-directed)





Significance randomization test for a correlation / causality measure q,
$H_0 : q = 0 \qquad H_1 : q \neq 0$

1. Generate M resampled (surrogate) time series, each by shifting the original observations with a random time step w:

   original time series: $\{x_t\} = \{x_1, x_2, \ldots, x_n\}$

   i-th surrogate time series:

   $\{x_t^{*i}\} = \{x_{w+1}, x_{w+2}, \ldots, x_n, x_1, \ldots, x_{w-1}, x_w\}$

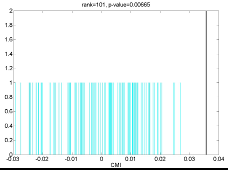
2. Compute the statistic q on the original pair, q_0, and on the M surrogate pairs, $q_1, \ldots, q_M$,

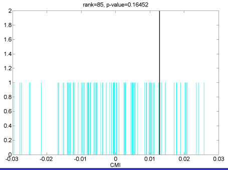
   e.g. $q_0 \equiv r_{XY}(\tau) = \mathrm{Corr}(x_t, y_{t+\tau})$ and $q_i \equiv \mathrm{Corr}(x_t^{*i}, y_{t+\tau}^{*i})$

3. If q_0 is at the tails of the empirical null distribution formed by $q_1, \ldots, q_M$, reject H_0.

   Using rank ordering: for a two-sided test, the p-value of the test is

$$
\begin{array}{lll}
2\frac{r_{q_0}-0.326}{M+1+0.348} & \text{if} & r_{q_0} < \frac{M+1}{2} \\
2\left(1-\frac{r_{q_0}-0.326}{M+1+0.348}\right) & \text{if} & r_{q_0} \geq \frac{M+1}{2}
\end{array}
$$





Example: Returns for USA, UnitedKingdom, Greece and Australia.
Correlation matrix for delay 1, $r_{XY}(1)$

$$R(1) = \begin{bmatrix} & 0.382 & 0.333 & 0.596 \\ 0.049 & & 0.039 & 0.303 \\ 0.096 & 0.001 & & 0.190 \\ 0.031 & -0.001 & -0.021 & \end{bmatrix}$$

Randomization significance test for $r_{XY}(1)$ ($M = 1000$)

Matrix of p-values   Adjacency matrix

$$P(R(1)) = \begin{bmatrix} & \color{red}{0.0013} & \color{red}{0.0013} & \color{red}{0.0033} \\ 0.0732 & & 0.1991 & \color{red}{0.0013} \\ \color{red}{0.0073} & 0.8901 & & \color{red}{0.0033} \\ 0.2450 & 0.9760 & 0.4028 & \end{bmatrix} A = \begin{bmatrix} & 1 & 1 & 1 \\ 0 & & 0 & 1 \\ 1 & 0 & & 1 \\ 0 & 0 & 0 & \end{bmatrix}$$

For significance level, say $\alpha = 0.05$, there may be $p < \alpha$ more often than it should be due to multiple testing.
Correction with e.g. False Discovery Rate (FDR)

Linear causality measures (direct and indirect)

Idea of Granger causality $X \to Y$ [Granger 1969]:
predict Y better when including X in the regression model.

Granger Causality Index (GCI) [Brandt & Williams 2007]

Bivariate time series $\{x_t, y_t\}_{t=1}^{n}$
driving system: X, response system: Y

Model 1 (restricted, R, X absent in the model):

$$y_t = \sum_{i=1}^{p} a_i y_{t-i} + e_{R,t}$$

Model 2 (unrestricted, U, X present in the model):

$$y_t = \sum_{i=1}^{p} a_i y_{t-i} + \sum_{i=1}^{p} b_i x_{t-i} + e_{U,t}$$

$$\mathrm{GCI}_{X\to Y} = \ln \frac{\mathrm{Var}(\hat{e}_{R,t})}{\mathrm{Var}(\hat{e}_{U,t})} \qquad \mathrm{GCI}_{X\to Y} > 0 \Rightarrow X \to Y \text{ holds}$$

Parametric significance test for GCI

$\mathrm{GCI}_{X \to Y} > 0$? $\Rightarrow$ Significance test

If X does not Granger causes Y then the contribution of X-lags in the unrestricted model should be insignificant $\Rightarrow$
the terms of X should be insignificant

H_0: $b_i = 0$, for all $i = 1, \ldots, p$
H_1: $b_i \neq 0$, for any of $i = 1, \ldots, p$

Snedecor-Fisher test (F-test):

$$F = \frac{(\mathrm{SSE}^R - \mathrm{SSE}^U)/p}{\mathrm{SSE}^U/\mathrm{ndf}}$$

SSE: sum of squared errors
ndf: number of degrees of freedoms, $\mathrm{ndf} = (n-p) - 2p$,
$n - p$: number of equations,
$2p$: number of coefficients in the U-model.

Linear causality measures (direct)

Conditional Granger Causality Index (CGCI)

K time series $\{x_t, y_t\}_{t=1}^n$ and $\{\mathbf{z}_t\}_{t=1}^n = \{z_{1,t}, z_{2,t}, \ldots, z_{K-2,t}\}_{t=1}^n$
driving system: X, response system: Y,
conditioning on system Z, $Z = \{Z_1, Z_2, \ldots, Z_{K-2}\}$

Model 1 (restricted, R, X absent in the model):

$$y_t = \sum_{i=1}^{p} a_i y_{t-i} + \sum_{i=1}^{p} A_i \mathbf{z}_{t-i} + e_{R,t}$$

Model 2 (unrestricted, U, X present in the model):

$$y_t = \sum_{i=1}^{p} a_i y_{t-i} + \sum_{i=1}^{p} b_i x_{t-i} + \sum_{i=1}^{p} A_i \mathbf{z}_{t-i} + e_{U,t}$$

$$\mathrm{CGCI}_{X \to Y|Z} = \ln \frac{\mathrm{Var}(\hat{e}_{R,t})}{\mathrm{Var}(\hat{e}_{U,t})}$$

Parametric significance test for CGCI

$\text{CGCI}_{X\to Y|Z} > 0$? $\Rightarrow$ Significance test as for GCI

H_0: $b_i = 0$, for all $i = 1,\ldots,p$
H_1: $b_i \neq 0$, for any of $i = 1,\ldots,p$

$$F = \frac{(\text{SSE}^R - \text{SSE}^U)/p}{\text{SSE}^U/\text{ndf}}$$

$\text{ndf} = (n-p) - Kp$,
$n-p$: number of equations,
Kp: number of coefficients in the U-model.

VAR model for Y

$$y_t = \sum_{i=1}^{p} a_i y_{t-i} + \sum_{i=1}^{p} b_i x_{t-i} + e_{U,t}$$

Model order and embedding parameters

VAR model for Y

$$y_t = \sum_{i=1}^{p} a_i y_{t-i} + \sum_{i=1}^{p} b_i x_{t-i} + e_{U,t}$$

$$y_{t+1} = \sum_{i=1}^{p} a_i y_{t-i+1} + \sum_{i=1}^{p} b_i x_{t-i+1} + e_{U,t+1}$$

y_{t+1} is given in terms of $\{y_t, y_{t-1}, \ldots, y_{t-p+1}\}$ and $\{x_t, x_{t-1}, \ldots, x_{t-p+1}\}$.

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$\mathbf{y}_t = [y_t, y_{t-1}, \ldots, y_{t-p+1}]$: vector of lagged Y

let the lag step be $\tau \geq 1 \quad \Rightarrow \quad \mathbf{y}_t = [y_t, y_{t-\tau}, \ldots, y_{t-(p-1)\tau}]$:

τ, p: embedding parameters (generally different for X and Y)

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τ, p: embedding parameters (generally different for X and Y)

State space reconstruction:

$\mathbf{x}_t = [x_t, x_{t-\tau_x}, \ldots, x_{t-(m_x-1)\tau_x}]'$, embedding parameters: m_x, τ_x
$\mathbf{y}_t = [y_t, y_{t-\tau_y}, \ldots, y_{t-(m_y-1)\tau_y}]'$, embedding parameters: m_y, τ_y

y_{t+1}: future state of Y

Nonlinear causality measures (direct and indirect)

Transfer Entropy (TE) [Schreiber, 2000]

Measure the effect of X on Y at one time step ahead, accounting (conditioning) for the effect from its own current state

$$
\begin{aligned}
\mathrm{TE}_{X\to Y} &= I(y_{t+1};\mathbf{x}_t|\mathbf{y}_t) \\
&= H(\mathbf{x}_t,\mathbf{y}_t) - H(y_{t+1},\mathbf{x}_t,\mathbf{y}_t) + H(y_{t+1},\mathbf{y}_t) - H(\mathbf{y}_t) \\
&= \sum p(y_{t+1},\mathbf{x}_t,\mathbf{y}_t)\log\frac{p(y_{t+1}|\mathbf{x}_t,\mathbf{y}_t)}{p(y_{t+1}|\mathbf{y}_t)}
\end{aligned}
$$

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Joint entropies (and distributions) can have high dimension!

Entropy estimates from nearest neighbors [Kraskov et al, 2004]

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Measure the effect of X on Y at one time step ahead, accounting (conditioning) for the effect from its own current state

$$\begin{aligned}
\mathrm{TE}_{X \to Y} &= I(y_{t+1}; \mathbf{x}_t | \mathbf{y}_t) \\
&= H(\mathbf{x}_t, \mathbf{y}_t) - H(y_{t+1}, \mathbf{x}_t, \mathbf{y}_t) + H(y_{t+1}, \mathbf{y}_t) - H(\mathbf{y}_t) \\
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\end{aligned}$$

Joint entropies (and distributions) can have high dimension!

Entropy estimates from nearest neighbors [Kraskov et al, 2004]

TE is equivalent to GCI when the stochastic process of (X, Y) is Gaussian [Barnett et al, PRE 2009]

Nonlinear causality measures (direct)

driving system: X, response system: Y,
conditioning on system Z, $Z = \{Z_1, Z_2, \ldots, Z_{K-2}\}$
join all $K-2$ z-reconstructed vectors: $\mathbf{Z}_t = [\mathbf{z}_{1,t}, \ldots, \mathbf{z}_{K-2,t}]$

Partial Transfer Entropy (PTE) [Vakorin et al, 2009; Papana et al, 2012]

Measure the effect of X on Y at T times ahead, accounting (conditioning) for the effect from its own current state and the current state of the other variables except X.

$$\mathrm{PTE}_{X\rightarrow Y|Z} = I(y_{t+1}; \mathbf{x}_t | \mathbf{y}_t, \mathbf{Z}_t)$$

$$= H(\mathbf{x}_t, \mathbf{y}_t | \mathbf{Z}_t) - H(y_{t+1}, \mathbf{x}_t, \mathbf{y}_t | \mathbf{Z}_t) + H(y_{t+1}, \mathbf{y}_t | \mathbf{Z}_t) - H(\mathbf{y}_t | \mathbf{Z}_t)$$

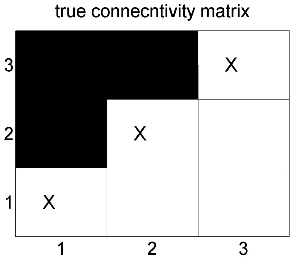
Joint entropies (and distributions) can have very high dimension!

Example: Nonlinear stochastic process

Nonlinear stochastic map:

$$
\begin{aligned}
x_{1,t} &= 3.4x_{1,t-1}(1 - x_{1,t-1}^2)e^{-x_{1,t-1}^2} + 0.4e_{1,t} \\
x_{2,t} &= 3.4x_{2,t-1}(1 - x_{2,t-1}^2)e^{-x_{2,t-1}^2} + 0.5x_{1,t-1}x_{2,t-1} + 0.4e_{2,t} \\
x_{3,t} &= 3.4x_{3,t-1}(1 - x_{3,t-1}^2)e^{-x_{3,t-1}^2} + 0.3x_{2,t-1} + 0.5x_{1,t-1}^2 + 0.4e_{3,t}
\end{aligned}
$$

[Model 7, Gourevich et al, 2006]



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\end{aligned}
$$

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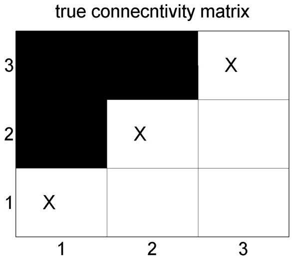
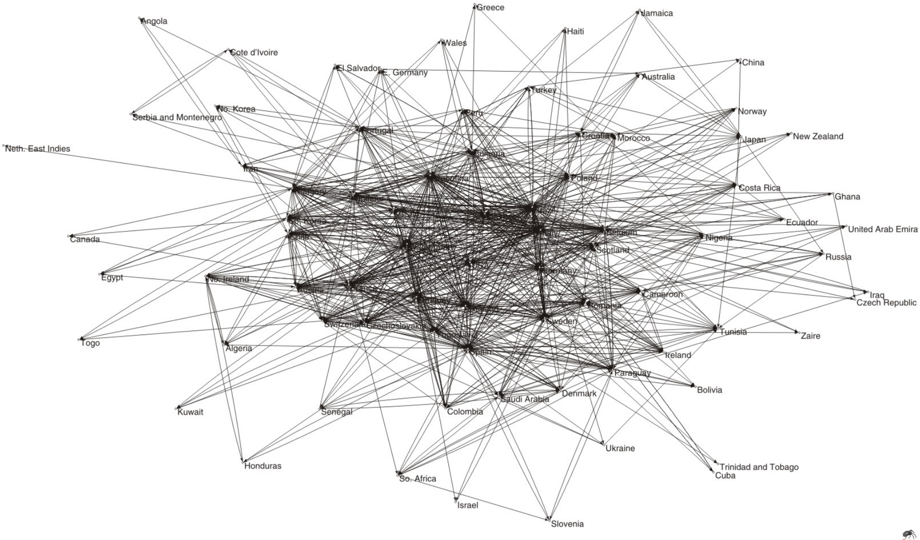


Estimation of the correct causality effects from the time series?

1. Dependence measures in univariate time series
2. Interdependence in multivariate time series
3. Complex networks from multivariate time series
4. High-dimensional time series: Implications and solutions

Example: Games of world cup 1930 - 2006



Example: Flight connections

Source: `https://au.pinterest.com/pin/488077678338752549/`

Example: Flight connections

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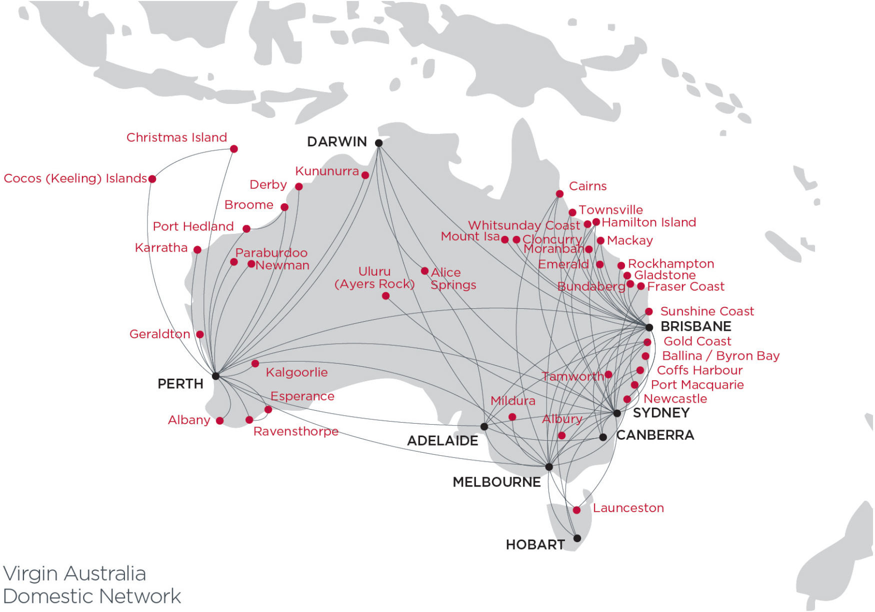
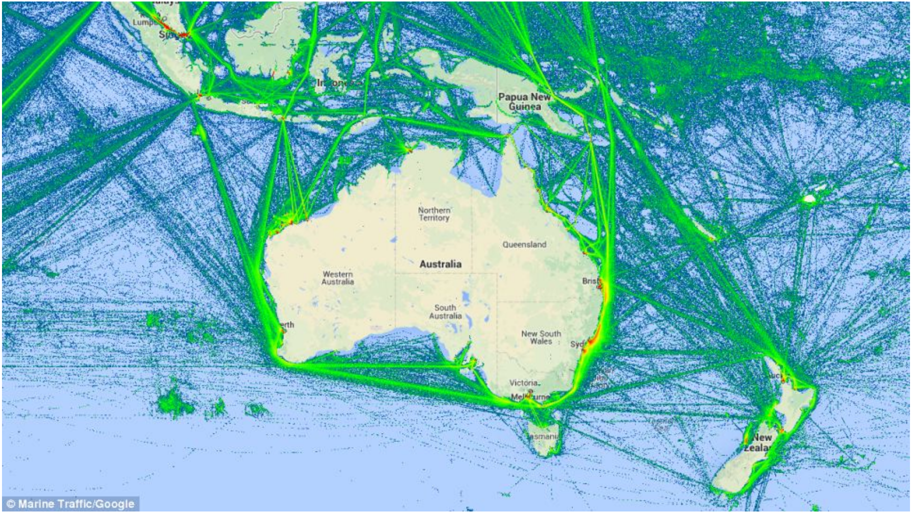


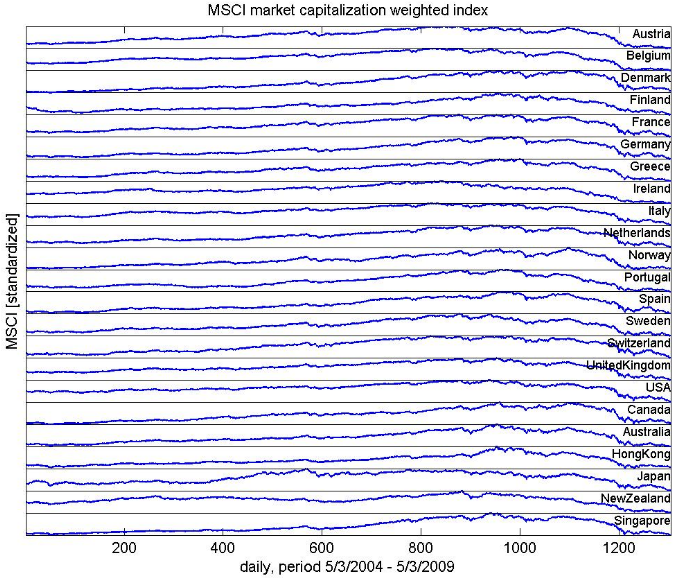
Example: Ship connections



Source:

http://i.dailymail.co.uk/i/pix/2014/05/22/article-2636152-1E1A482300000578-442_964x541.jpg

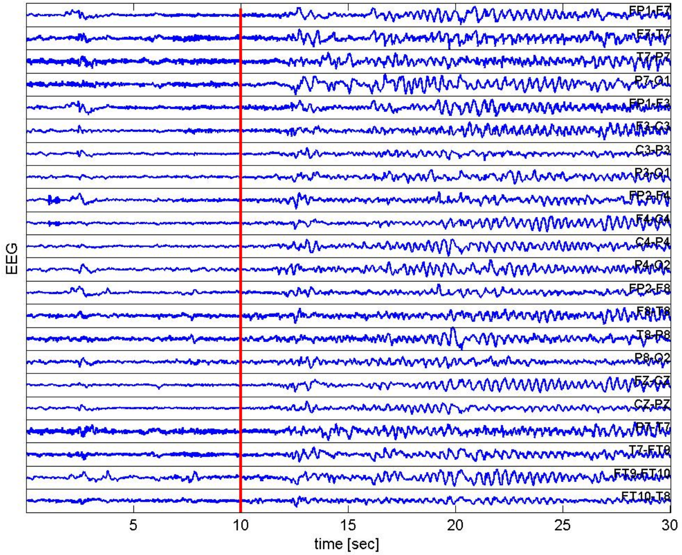
Example: Global financial market



Data source: `https://www.msci.com/market-cap-weighted-indexes`

Network ?

Example: Brain dynamical system



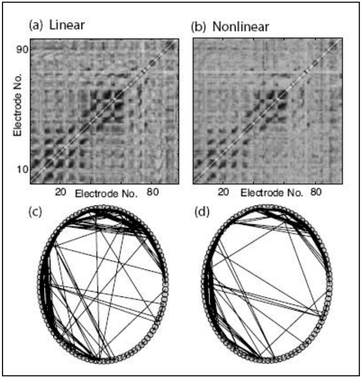
Data source: `https://physionet.org/pn6/chbmit/chb08/`

Network ?

PHYSICAL REVIEW E **79**, 061916 (2009)

Network inference with confidence from multivariate time series

Mark A. Kramer,[1,*] Uri T. Eden,[1] Sydney S. Cash,[2] and Eric D. Kolaczyk[1]
[1]*Department of Mathematics and Statistics, Boston University, Boston, Massachusetts 02215, USA*
[2]*Department of Neurology, Epilepsy Service, Harvard Medical School, ACC 835, Massachusetts General Hospital, 55 Fruit Street, Boston, Massachusetts 02114, USA*
(Received 9 March 2009; revised manuscript received 14 May 2009; published 11 June 2009)



Example: brain network

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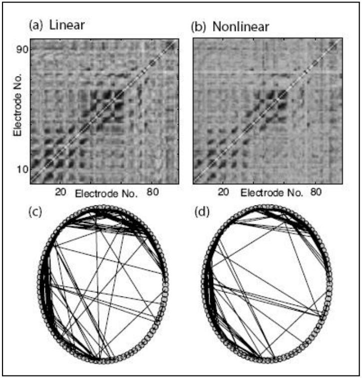

ECoG: "Linear and nonlinear association measures produce similar association matrices and networks."

Example: brain network

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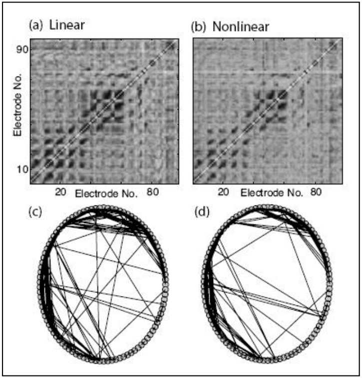

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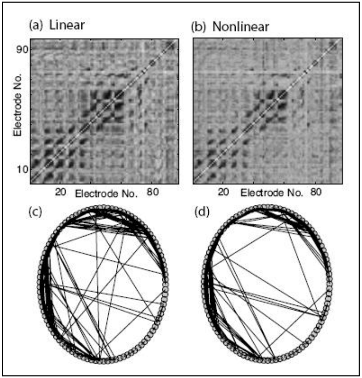

ECoG: "Linear and nonlinear association measures produce similar association matrices and networks."

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It is important to:

- Use appropriate associate measure $\Rightarrow$ weighted connection

Example: brain network

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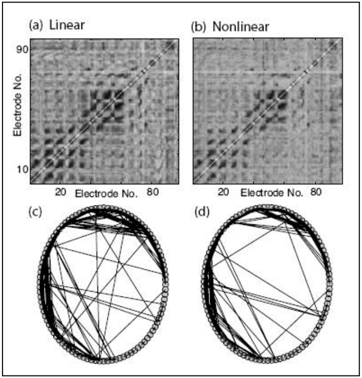

ECoG: "Linear and nonlinear association measures produce similar association matrices and networks."
?

It is important to:

- Use appropriate associate measure $\Rightarrow$ weighted connection
- Assess the significance of the measure $\Rightarrow$ binary connection

Example: World financial markets

$N = 8$ world stock markets, daily indices, $n = 100$ days.

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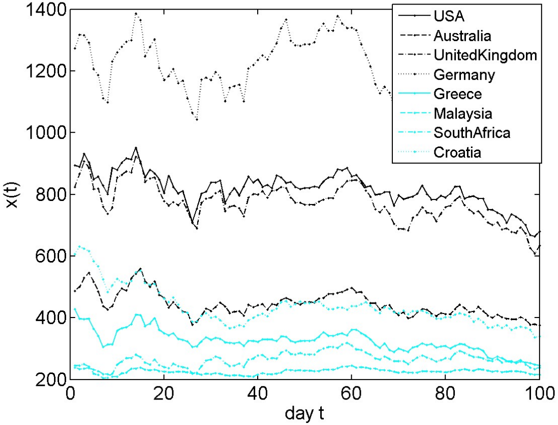

Example: World financial markets

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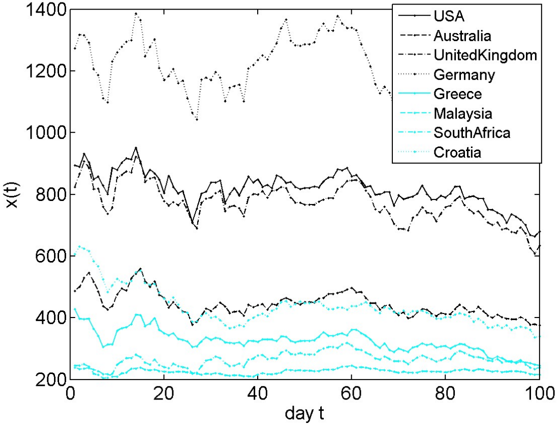

Similar indices, links among world stock markets?

Example: World financial markets, correlation coefficient

Upper triangular: sample correlation coefficient r_{ij}.
Lower triangular: p-value for significance test for ρ_{ij} (z-statistic)

	USA	AUS	UK	GER	GRE	MAL	SAF	CRO
USA		0.86	0.92	0.88	0.89	0.33	0.27	0.75
AUS	0		0.91	0.82	0.90	0.56	0.27	0.83
UK	0	0		0.88	0.92	0.40	0.31	0.74
GER	0	0	0		0.84	0.44	0.53	0.61
GRE	0	0	0	0		0.40	0.16	0.82
MAL	0.0008	0	0	0	0		0.54	0.38
SAF	0.0057	0.0065	0.0017	0	0.1154	0		-0.15
CRO	0	0	0	0	0	0.0001	0.1408	

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UK	0	0		0.88	0.92	0.40	0.31	0.74
GER	0	0	0		0.84	0.44	0.53	0.61
GRE	0	0	0	0		0.40	0.16	0.82
MAL	0.0008	0	0	0	0		0.54	0.38
SAF	0.0057	0.0065	0.0017	0	0.1154	0		-0.15
CRO	0	0	0	0	0	0.0001	0.1408	

Almost all indices are strongly correlated.

Example: World financial markets, correlation network

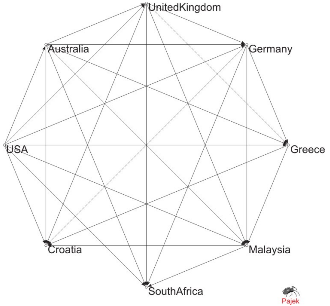
Adjacency matrix, threshold at $\alpha = 0.01$ (multiple testing?)

	USA	AUS	UK	GER	GRE	MAL	SAF	CRO
USA	0	1	1	1	1	1	1	1
AUS	1	0	1	1	1	1	1	1
UK	1	1	0	1	1	1	1	1
GER	1	1	1	0	1	1	1	1
GRE	1	1	1	1	0	1	0	1
MAL	1	1	1	1	1	0	1	1
SAF	1	1	1	1	0	1	0	0
CRO	1	1	1	1	1	1	0	0

Example: World financial markets, correlation network

Adjacency matrix, threshold at $\alpha = 0.01$ (multiple testing?)

	USA	AUS	UK	GER	GRE	MAL	SAF	CRO
USA	0	1	1	1	1	1	1	1
AUS	1	0	1	1	1	1	1	1
UK	1	1	0	1	1	1	1	1
GER	1	1	1	0	1	1	1	1
GRE	1	1	1	1	0	1	0	1
MAL	1	1	1	1	1	0	1	1
SAF	1	1	1	1	0	1	0	0
CRO	1	1	1	1	1	1	0	0



Example: World financial markets, partial correlation

Upper triangular: partial correlation $r_{ij|K}$, conditioned on all $|K| = 6$ rest variables.
Lower triangular: p-value for significance test for $\rho_{ij|K}$ (z-statistic)

	USA	AUS	UK	GER	GRE	MAL	SAF	CRO
USA		0.01	0.37	0.27	0.07	-0.27	0.11	0.27
AUS	0.9378		0.42	-0.02	0.15	0.30	0.10	0.38
UK	0.0002	0		0.08	0.36	-0.16	0.08	-0.11
GER	0.0081	0.8469	0.4693		0.38	-0.31	0.66	0.26
GRE	0.4946	0.1392	0.0003	0.0001		0.19	-0.36	0.01
MAL	0.0083	0.0033	0.1232	0.0026	0.0710		0.68	0.46
SAF	0.2908	0.3554	0.4321	0	0.0003	0		-0.70
CRO	0.0079	0.0002	0.3149	0.0099	0.9083	0	0	

Example: World financial markets, partial correlation

Upper triangular: partial correlation $r_{ij|K}$, conditioned on all $|K| = 6$ rest variables.
Lower triangular: p-value for significance test for $\rho_{ij|K}$ (z-statistic)

	USA	AUS	UK	GER	GRE	MAL	SAF	CRO
USA		0.01	0.37	0.27	0.07	-0.27	0.11	0.27
AUS	0.9378		0.42	-0.02	0.15	0.30	0.10	0.38
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Correlation between any two indices decreased when conditioned on all others.

Example: Financial markets, partial correlation network

Adjacency matrix, threshold at $\alpha = 0.01$

	USA	AUS	UK	GER	GRE	MAL	SAF	CRO
USA	0	0	1	1	0	1	0	1
AUS	0	0	1	0	0	1	0	1
UK	1	1	0	0	1	0	0	0
GER	1	0	1	0	1	1	1	1
GRE	0	0	1	1	0	0	1	0
MAL	1	1	1	1	0	0	1	1
SAF	0	0	1	1	1	1	0	1
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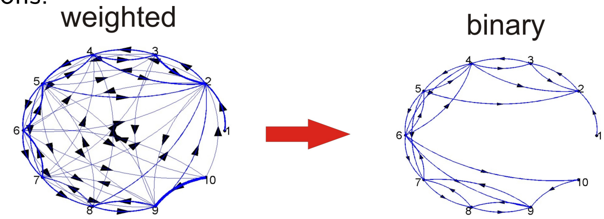
SAF	0	0	1	1	1	1	0	1
CRO	1	1	1	1	0	1	1	0

How to assess the presence of a connection?

Three possible ways to convert a network of weighted connections (the Granger causality measure) to a network of binary connections:



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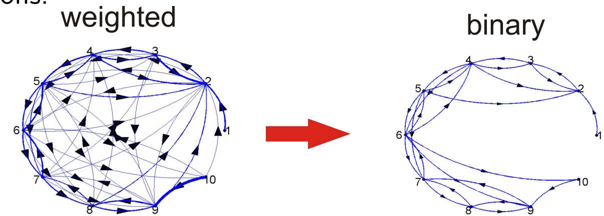

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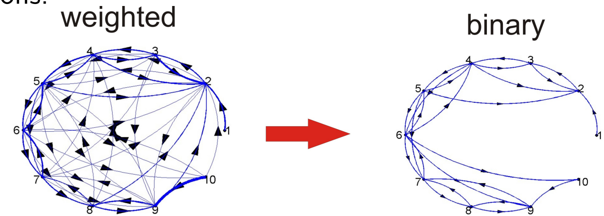

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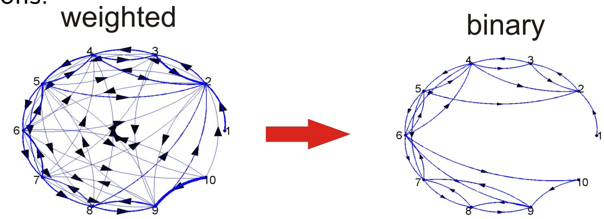

1. Threshold on the measure magnitude, $q(i \to j) > \text{thr}$.
2. Threshold on the network density, only the $d\%$ largest $q(i \to j)$.
3. Significance test on each $q(i \to j)$. Threshold, e.g. $\alpha = 0.05$ on the p-value of the test.
   Parametric or resampling test (resampling test for a nonlinear causality measure).

The problem of multiple testing

Significance resampling test on $q(i \to j)$ for each pair (X_i, X_j).

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False Discovery Rate (FDR) [Benjamini & Hochberg, 1995]

- $K(K-1)$ p-values in ascending order: $p_{(1)}, p_{(2)}, \ldots, p_{(K(K-1))}$
- Rejection for the k tests with $p \leq p_{(k)}$, where $p_{(k)}$ is the largest p-value for which $p_{(k)} < k\alpha/(K(K-1))$.

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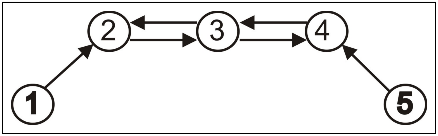
When K gets large, FDR requires **huge** M (impractical).

Example: coupled Henon maps

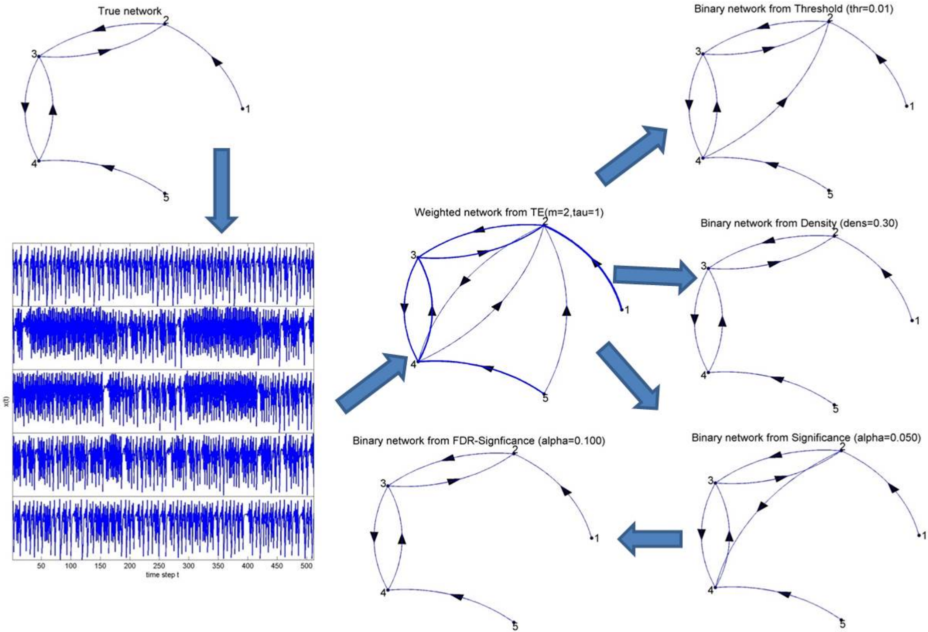
$$
\begin{aligned}
x_{1,t+1} &= 1.4 - x_{1,t}^2 + 0.3x_{1,t-1} \\
x_{i,t+1} &= 1.4 - (0.5C(x_{i-1,t} + x_{i+1,t}) + (1-C)x_{i,t})^2 + 0.3x_{i,t-1} \\
x_{K,t+1} &= 1.4 - x_{K,t}^2 + 0.3x_{K,t-1}
\end{aligned}
$$

C: coupling strength [Politi & Torcini, 1992]

Network structure for $K = 5$

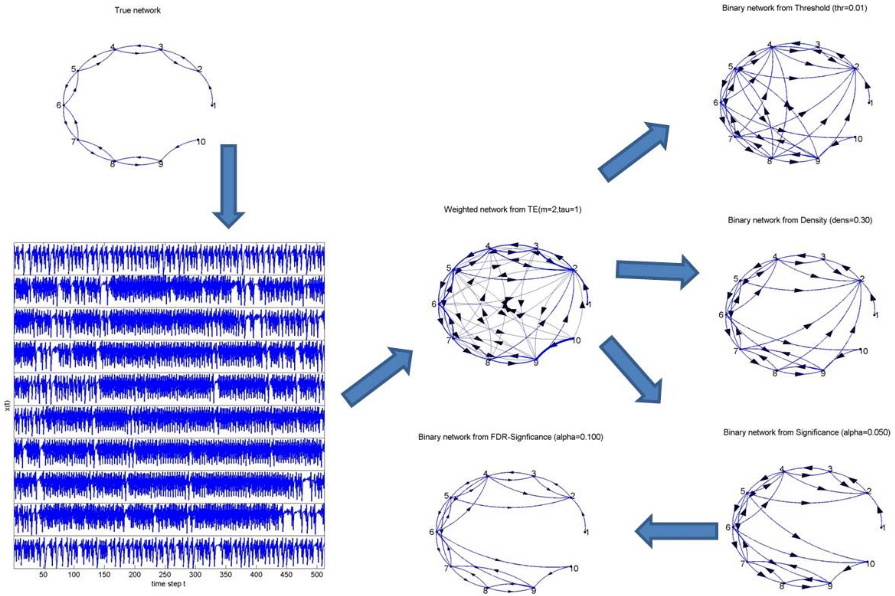


Example, TE, $K = 5$



1. Dependence measures in univariate time series
2. Interdependence in multivariate time series
3. Complex networks from multivariate time series
4. High-dimensional time series: Implications and solutions

Example, TE, $K = 10$



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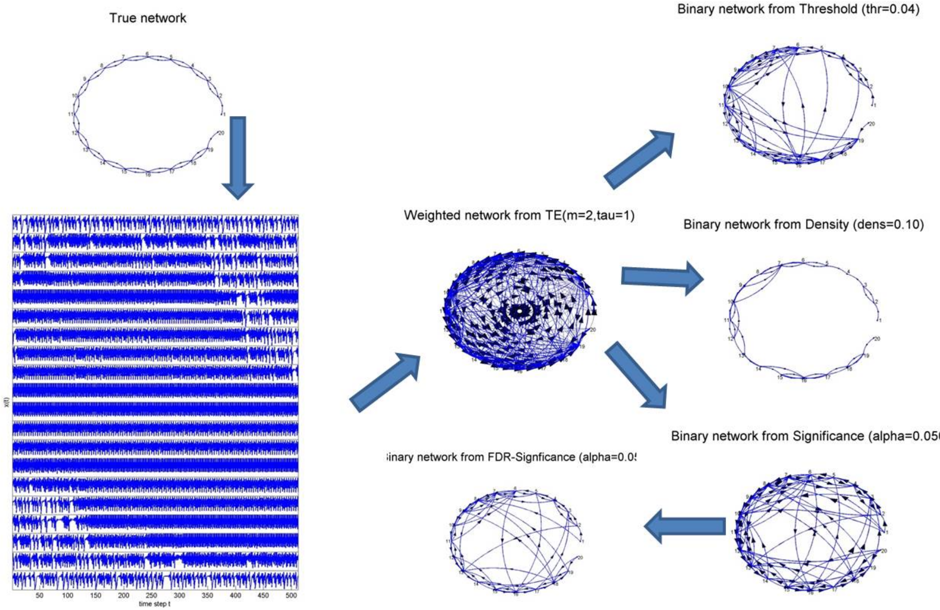
Example, TE, $K = 20$



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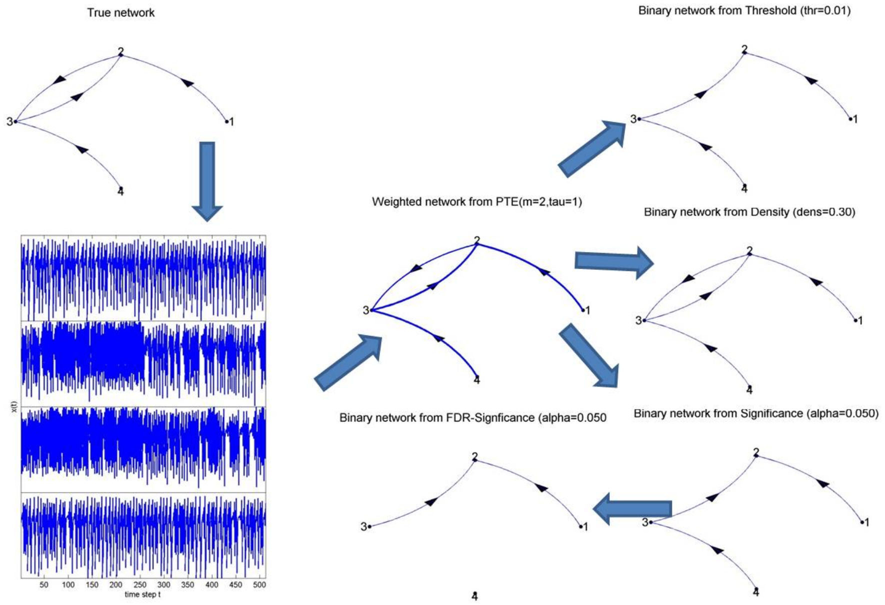
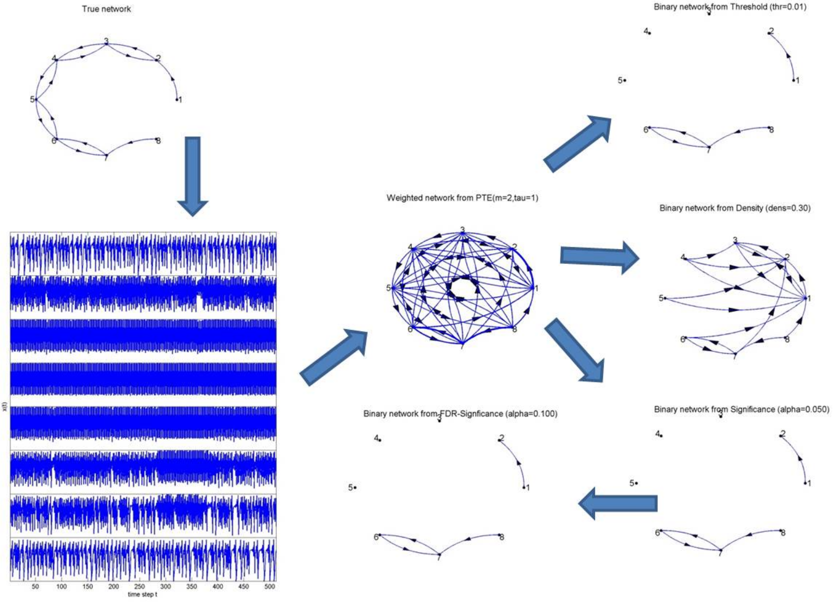
• Multivariate measures require long time series, e.g. $\mathrm{PTE}_{X \to Y|Z} = I(y_{t+1}; \mathbf{x}_t | \mathbf{y}_t, \mathbf{Z}_t)$ requires the estimation of entropy of $[y_{t+1}, \mathbf{x}_t, \mathbf{y}_t, \mathbf{Z}_t]'$ of dimension $1 + Km$.

Example, PTE, $K = 4$



Example, PTE, $K = 8$



Interdependence using Dimension Reduction

K time series $\{x_t, y_t\}_{t=1}^n$ and $\{\mathbf{z}_t\}_{t=1}^n = \{z_{1,t}, z_{2,t}, \dots, z_{K-2,t}\}_{t=1}^n$
driving system: X, response system: Y,
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In CGCI, the VAR model

$$y_t = \sum_{i=1}^{p} a_i y_{t-i} + \sum_{i=1}^{p} b_i x_{t-i} + \sum_{i=1}^{p} A_i \mathbf{z}_{t-i} + e_{U,t}$$

has Kp lagged variables.

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Statistics methodology: dimension reduction, sparse regression, restricted regression, and sparse/retricted VAR models.

Partial Mutual Information from Mixed Embedding (PMIME) applies dimension reduction using mutual information. The idea [Vlachos & Kugiumtzis, 2010]:

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Similar approaches based on this idea: [Faes et al, PRE 2011; Stamaglia et al, PRE 2012; Runge et al, PRL 2012; Wibral et al, PLOSOne 2013; Runge et al, PRE 2015; edited book of Wibral, Vicente and Lizier "Directed information measures in Neuroscience", Springer, 2014.]

PMIME - 2

The mixed embedding scheme

- Start with an empty embedding vector $\mathbf{w}_t^0$, future vector of Y, y_{t+1}, and maximum lag L (or L_x for X, L_y for Y etc) $\mathbf{W}_t = \{x_t, \ldots, x_{t-L-1}, y_t, \ldots, y_{t-L-1}, z_{1,t}, \ldots, z_{K-2,t-L-1}\}$

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- Progressive vector building stops at step j ($\mathbf{w}_t = \mathbf{w}_t^{j-1}$):

  Criterion of hard threshold:
  $I\left(y_{t+1}; \mathbf{w}_t^{j-1}\right) / I\left(y_{t+1}; \mathbf{w}_t^{j}\right) > A$ (here $A = 0.95$)

  Criterion of adaptive threshold:
  randomization significance test on $I(y_{t+1}; w_t^j | \mathbf{w}_t^{j-1})$

The non-uniform mixed embedding vector of lags of all X, Y, Z for explaining y_{t+1}:

$$\mathbf{w}_t = (\underbrace{x_{t-\tau_{x1}}, \ldots, x_{t-\tau_{xm_x}}}_{\mathbf{w}_t^x}, \underbrace{y_{t-\tau_{y1}}, \ldots, y_{t-\tau_{ym_y}}}_{\mathbf{w}_t^y}, \underbrace{z_{t-\tau_{z1}}, \ldots, z_{t-\tau_{zm_z}}}_{\mathbf{w}_t^z})$$

PMIME - 3

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The causality measure PMIME

$$R_{X \to Y|Z} = \frac{I(y_{t+1}; \mathbf{w}_t^x \mid \mathbf{w}_t^y, \mathbf{w}_t^Z)}{I(y_{t+1}; \mathbf{w}_t)}$$

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- If $\mathbf{w}_t$ contains no components from X, then $R_{X\to Y|Z} = 0$ and X has no direct effect on the future of Y.

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- $R_{X \to Y|Z} = 0$ when no significant causality is present, and $R_{X \to Y|Z} > 0$ when it is present
  [no significance test, no issues with multiple testing!]

PMIME - 4

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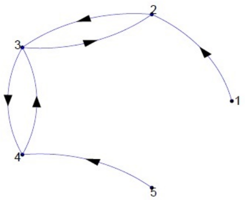
$\Rightarrow$ good candidate for causality analysis with many variables

Example: coupled Mackey-Glass

Coupled identical Mackey-Glass delayed differential equations

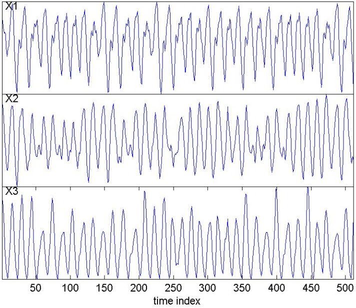
$$\dot{x}_i(t) = -0.1x_i(t) + \sum_{j=1}^{K} \frac{C_{ij}x_j(t-\Delta)}{1+x_j(t-\Delta)^{10}} \quad \text{for} \quad i = 1,\ldots,K$$

$K = 5$



Mackey-Glass, $C = 0.2$

$\Delta = 20$



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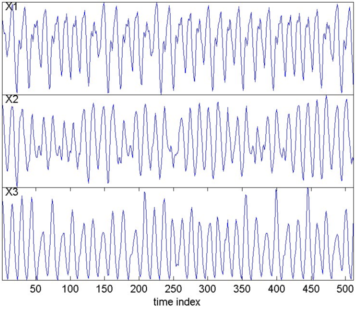
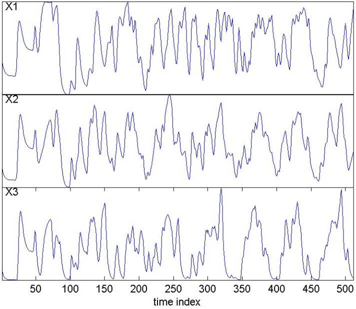
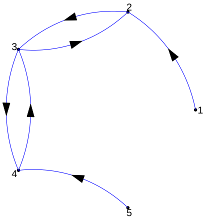
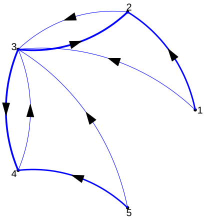


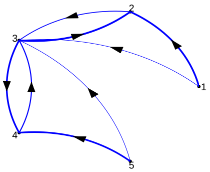
$\Delta = 100$



$K = 5$ True from PMIME ($\Delta = 20$) from PMIME ($\Delta = 100$)

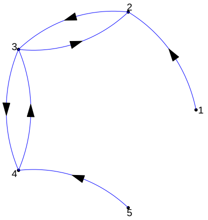


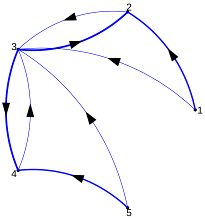


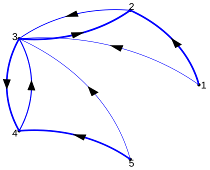


Mackey-Glass: true/estimated network [Kugiumtzis & Kimiskidis, 2015]

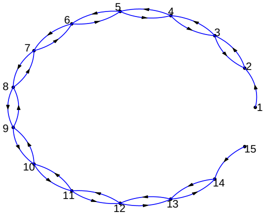
$K = 5$ True from PMIME ($\Delta = 20$) from PMIME ($\Delta = 100$)

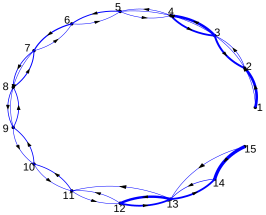


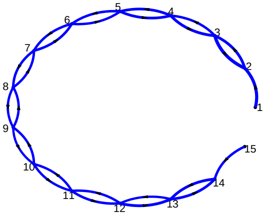




$K = 15$ True from PMIME ($\Delta = 20$) from PMIME ($\Delta = 100$)

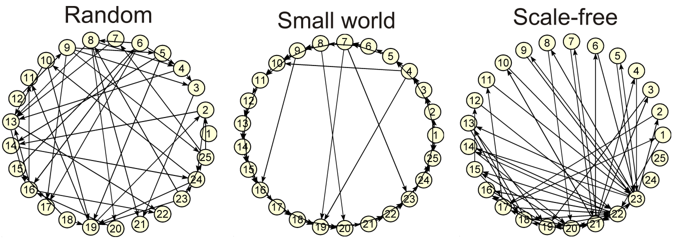






Can different network structures be detected?

Simulation: three types of networks for the generating system



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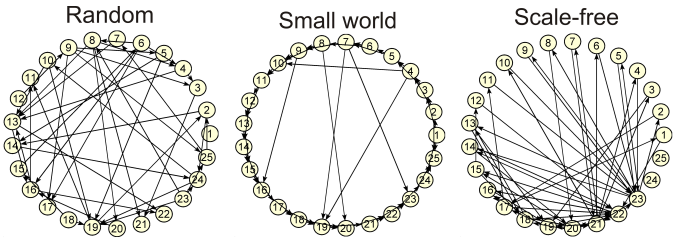
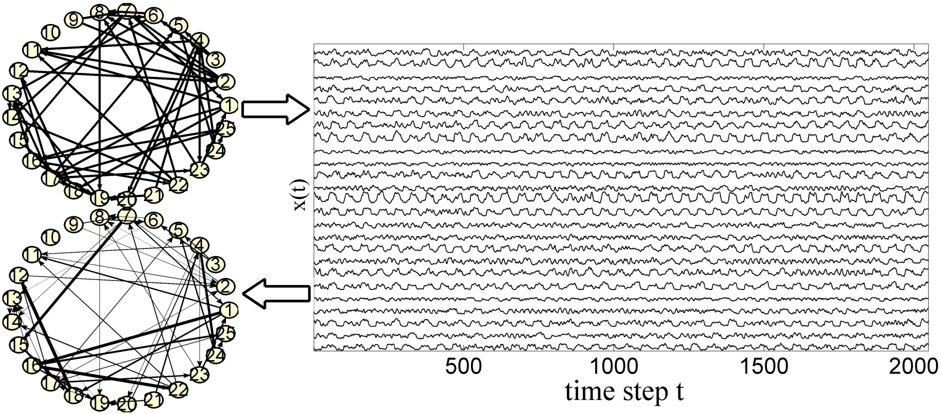


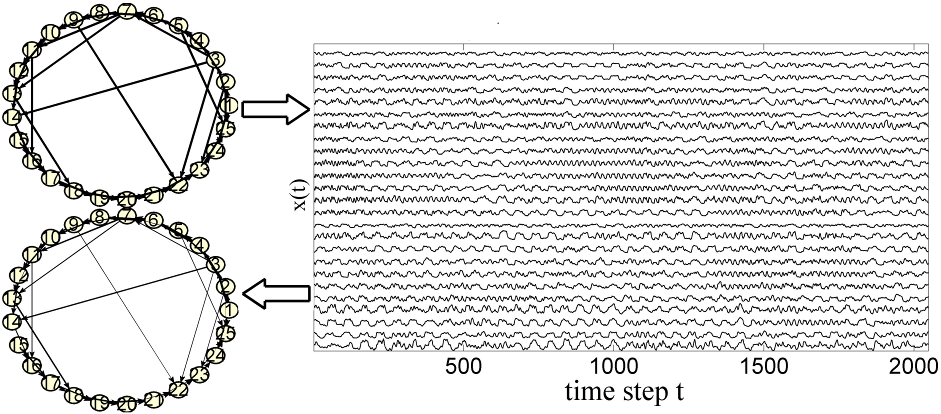
Generating system:
coupled Mackey-Glass system, $K = 25$, $\Delta = 100$, $C = 0.2$
with coupling structure defined by the network type

Causality measure: PMIME

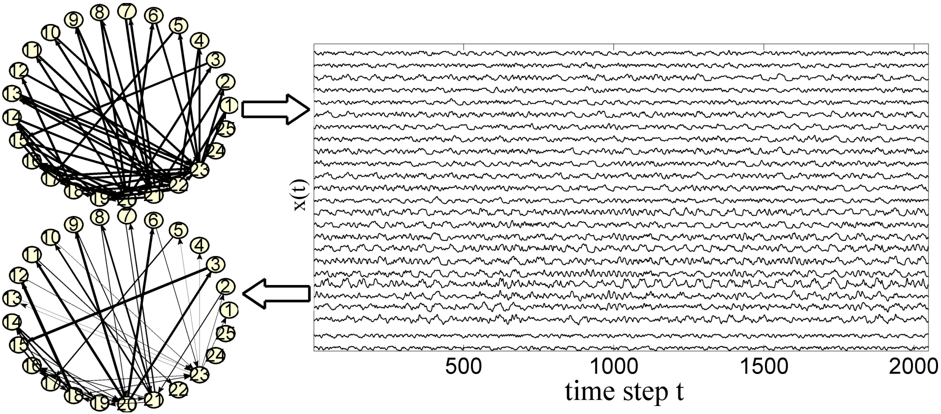
Estimation of the Random Network



Estimation of the Small-World Network



Estimation of the Scale-Free Network



Simulation example:

The network structure undergoes structural change at specific time points:
Random $\Rightarrow$ Small-World $\Rightarrow$ Scale-Free

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Estimation of networks with PMIME at sliding windows

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Estimation of network characteristics on the PMIME networks

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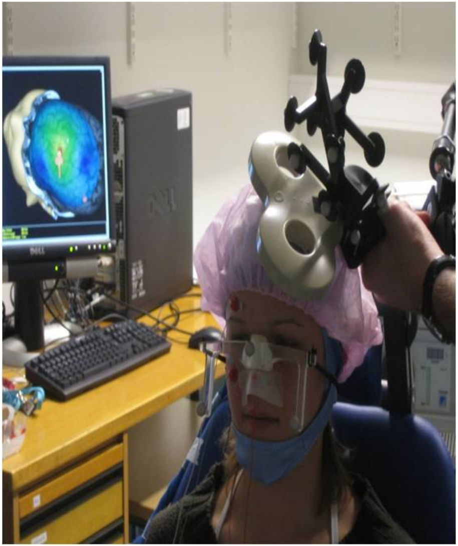
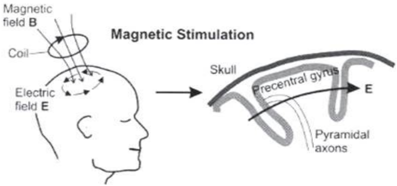
Estimation of networks with PMIME at sliding windows

Estimation of network characteristics on the PMIME networks

Structural change detection,
[Slow], **[Middle]**, **[Fast]**, **[Very fast]**



Jointly with Vassilis Kimiskidis,
Medical School, AUTh



How does TMS act on epileptic brain connectivity?

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$\{X_1, X_2, \ldots, X_K\}$: K EEG channels, each represents a (sub)system

Practical problems to overcome:

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- Application on small time windows $\Rightarrow$ limited data size

TMS-EEG: brain connectivity analysis

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- scalp EEG $\Rightarrow$ many channels $\Rightarrow$ many variables in Z to account for

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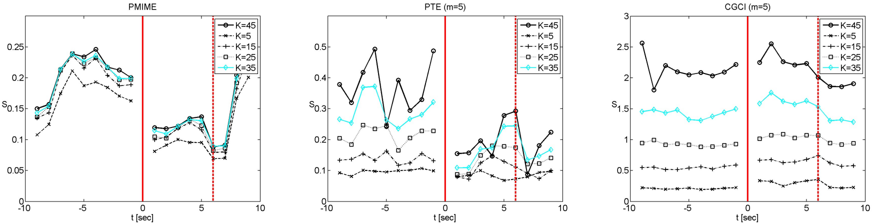
PMIME addresses all these problems!

[Kugiumtzis, PRE, 2013]

One epileptiform discharge (ED) episode terminated by transcranial magnetic stimulation (TMS), totally 45 channels

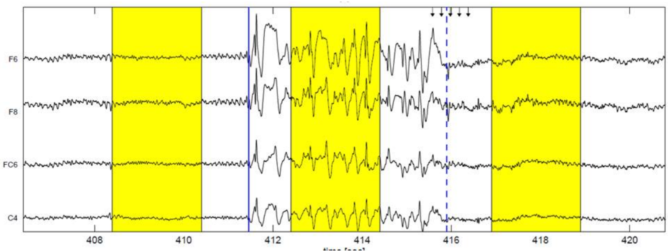
- Select randomly a subset of channels.
- Compute the connectivity measures on the subset at each sliding window
- Compute average connectivity strength at each sliding window.
- Repeat the steps above a number of times (here 12).

... for subsets of 5, 15, 25, 35 and once for 45 channels.



Preprocessing: **[One Episode]**

- replacement of TMS artifact, high order FIR
- rejection of channels with artifacts
- reference to infinity (REST) [Qin et al, ClinNeuroph 2010]
- overlapping windows of 2s, a sliding step of 0.5s



Epileptiform discharges induced by TMS

Preprocessing: **[One Episode]**

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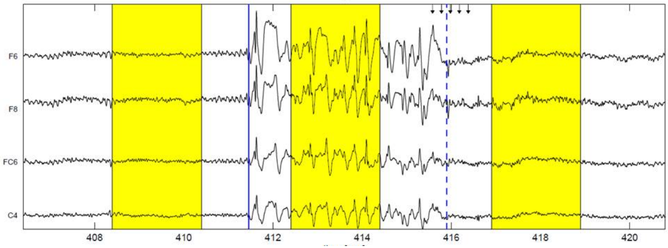
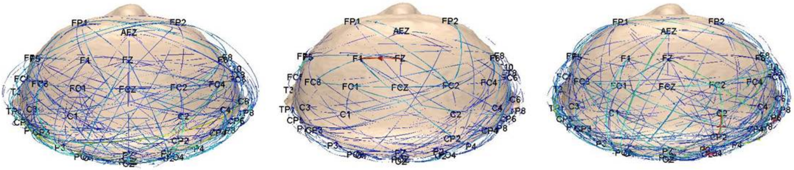


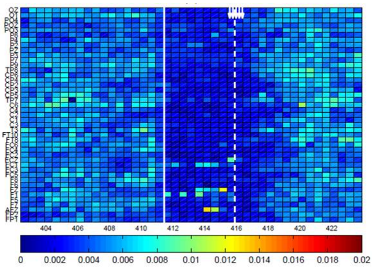


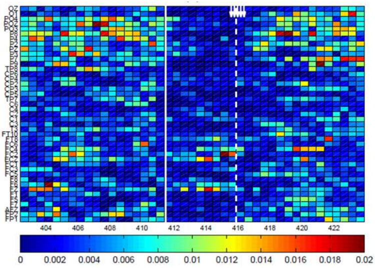
subject 1 with focal seizure, ED episode ends with TMS

In-Strength

Out-Strength

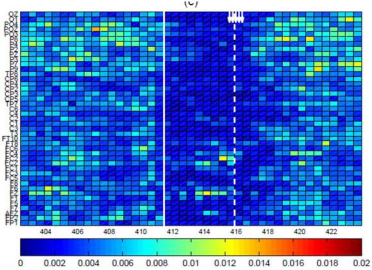
[Kugiumtzis & Kimiskidis, 2015]

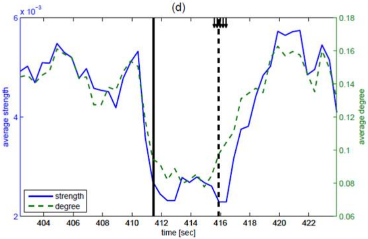




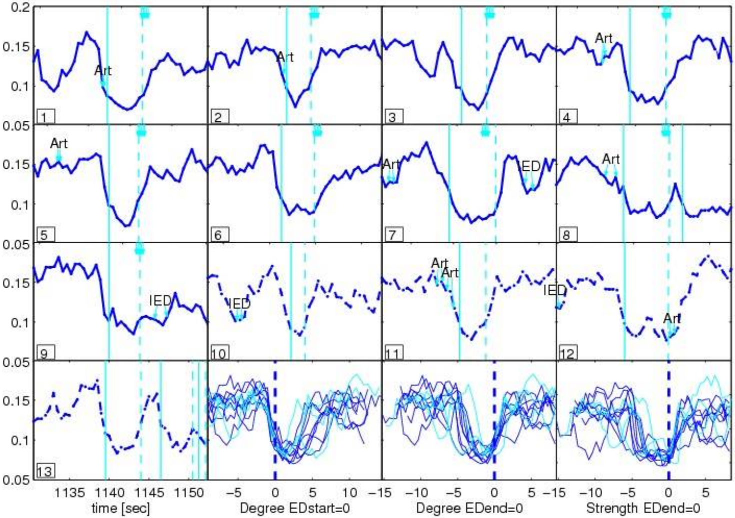
In-Out-Strength

Average strength / degree





Subject 1 with focal seizure, 13 episodes, average degree



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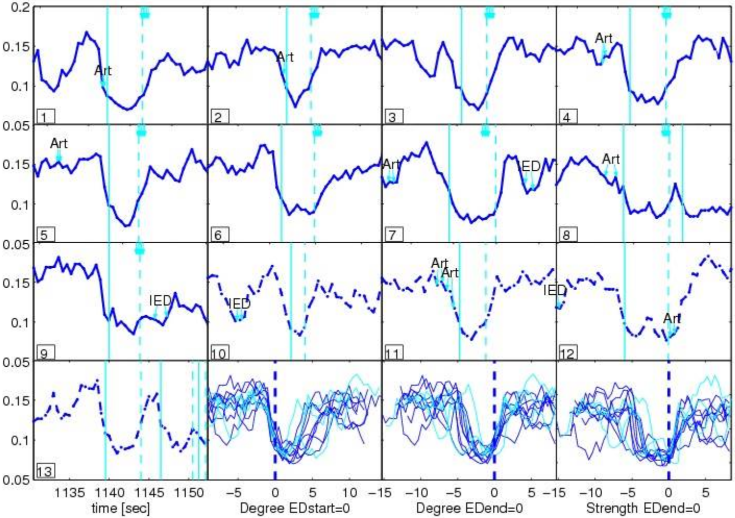
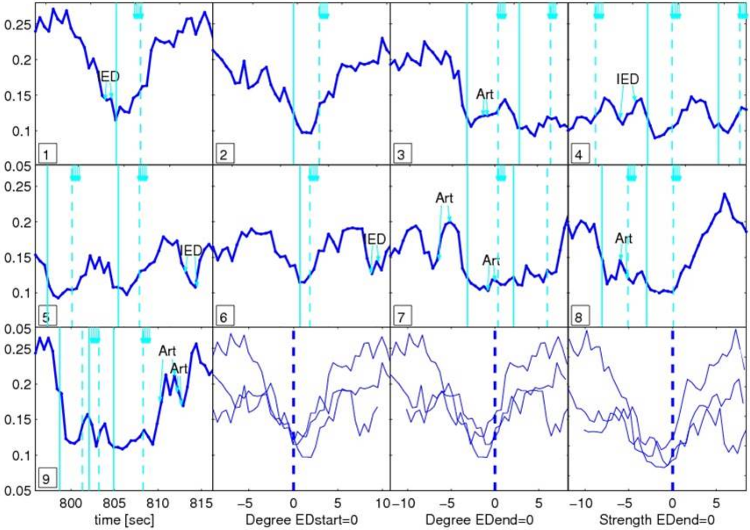


TMS terminates the ED prematurely and restores the network structure as if it would have terminated spontaneously

Subject 2 with focal seizure, 9 episodes, average degree



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Many network indices (totally 78) computed on the PMIME-causality networks

Symbol	Description
deg^m	degree distribution, m=mean,std,skewness,kurtosis
str^m	strength distribution, m=mean,std,skewness,kurtosis
TrR_k	transitivity ratio, k=binary undirected (bu),binary directed (bd) weighted directed (wd)
$EigC^m$	eigenvector centrality distribution, m=mean,std
λ_k	characteristic path length, k=bd,wd
GE_k	global efficiency, k=bd,wd
ϵ_k^m	eccentricity distribution, m=mean,std and k=bd,wd
rad_k	radius, k=bd,wd
d_k	diameter, k=bd,wd
C_k^m	clustering coefficient distribution,m=mean,std and k=bd,wd
g_k^m	betweenness centrality distribution,m=mean,std and k=bd,wd
$e-g_k^m$	edge betweenness centrality distribution,m=mean,std and k=bd,wd
LE_k^m	local efficiency distribution,m=mean,std and k=bd,wd
$3motif(i)$	ith motif of 3 nodes, i=1,2,...13
$modul(i)$	modularity for i modules, i=2,3,5
$r_{deg}(i,j)$	assortativity coefficient in terms of the degree, i=in,out and j=in,out or i,j=und
$r_{str}(i,j)$	assortativity coefficient in terms of the strength, i=in,out and j=in,out or i,j=und
p_{top}	Rent exponent:topological
p_{ph}	Rent exponent:physical
p_{ee}	Rent exponent:efficient embedding
SW_k	small-worldness, k=bd,wd
kcs	k-core size, k=90-percentile of degree distribution
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ϕ_k	Rich club coefficient, k=bd,wd
$cycprob_1$	fraction of 3-cycles out of 3-paths
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For both subjects and pairs
preED - ED
ED - postED

Measure	AUROC
deg^{mean}	0.9296
d_{bd}	0.9268
$3motif(1)$	0.9231
λ_{bd}	0.9231
str^{mean}	0.9207
$3motif(3)$	0.9206
$3motif(5)$	0.9199
LE_{bd}^{mean}	0.9173
GE_{wd}	0.9163
TrR_{wd}	0.9150

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Many networks indices discriminate well preED, ED, postED

ED induced by TMS, **[PMIME on 2s windows]** [Kugiumtzis et al, 2016]

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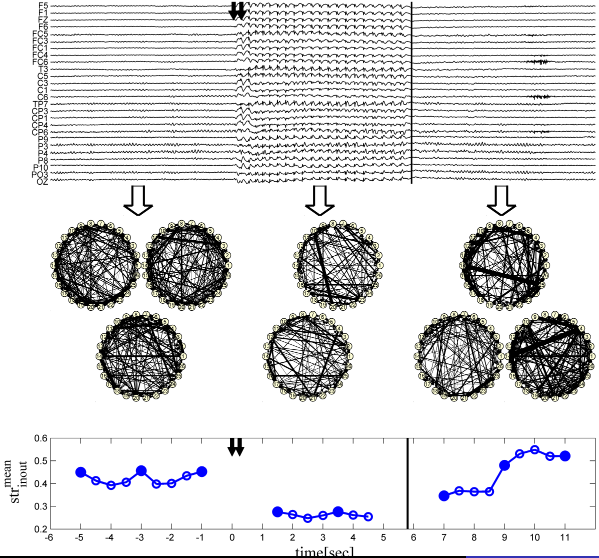
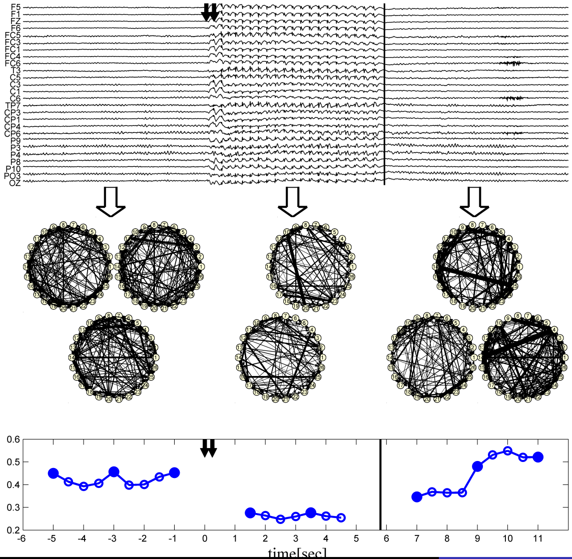
One episode



subject 1 with genetic generalized epilepsy (GGE)

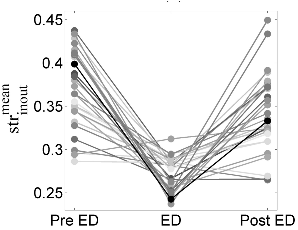
ED induced by TMS, **[PMIME on 2s windows]** [Kugiumtzis et al, 2016]

One episode

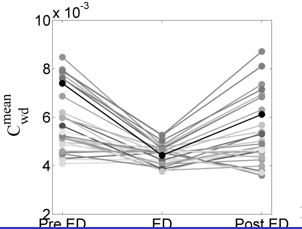


29 episodes

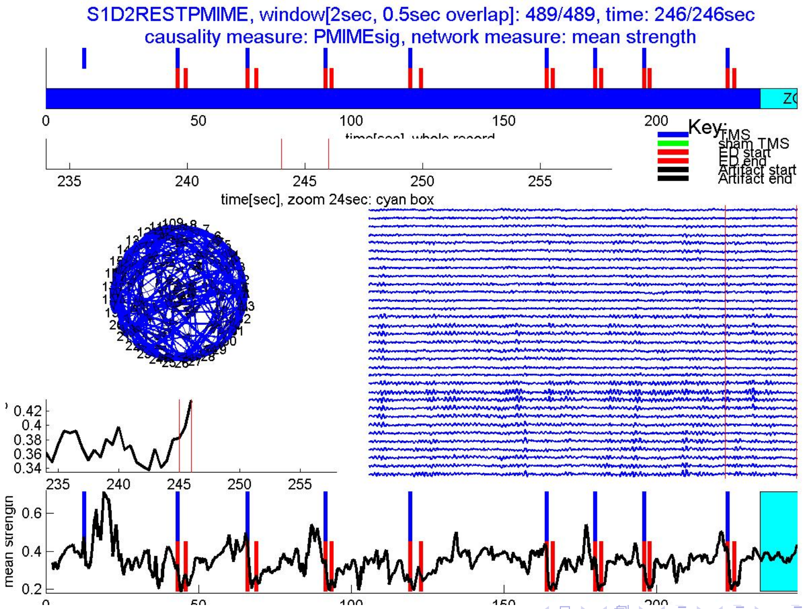
Average strength



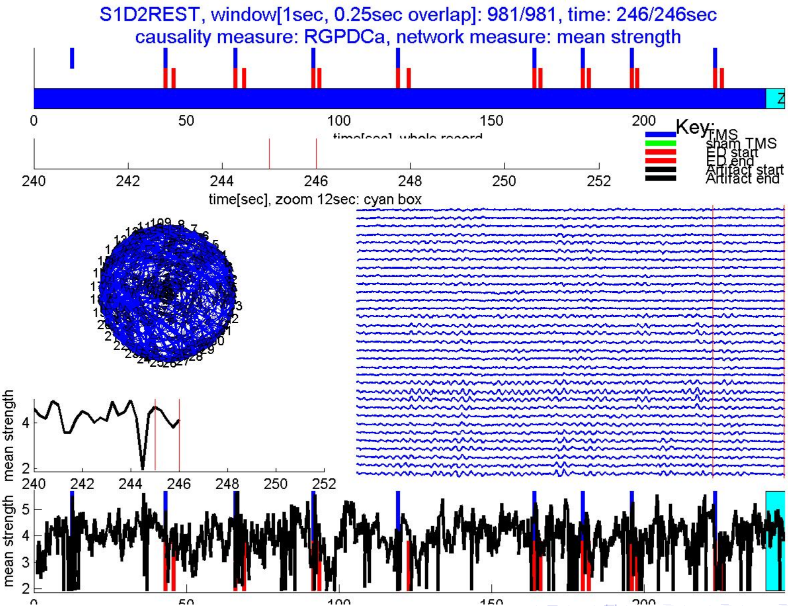
Average clustering coefficient



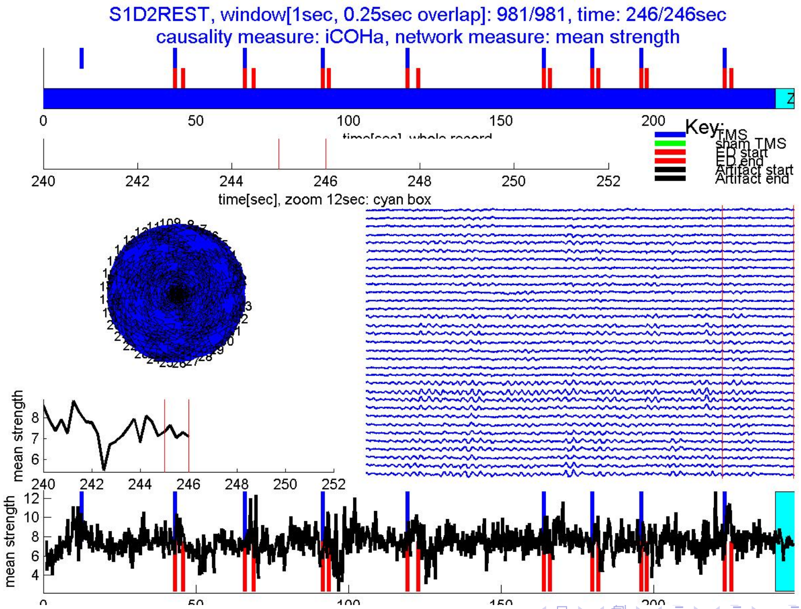
PMIME on 2s window



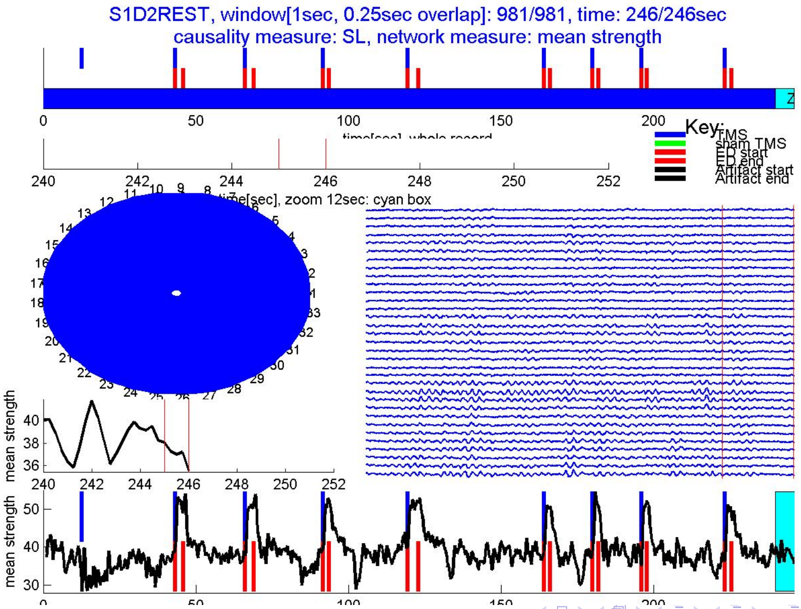
RGPDC(α) on 1s window



iCoh(α) on 1s window



Synchronization Likelihood (SL) on 1s window



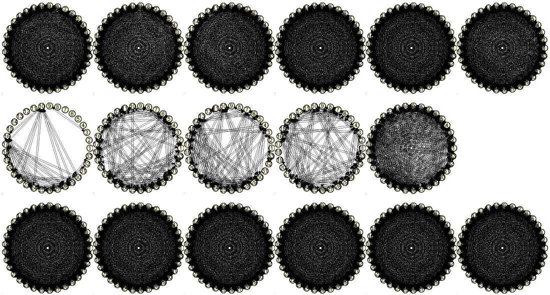
subject 1 with GGE

ED induced by TMS, SL on 1s windows,

subject 1 with GGE

ED induced by TMS, SL on 1s windows,

One episode



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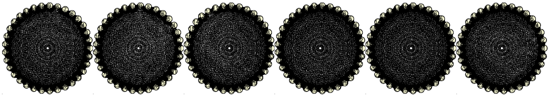
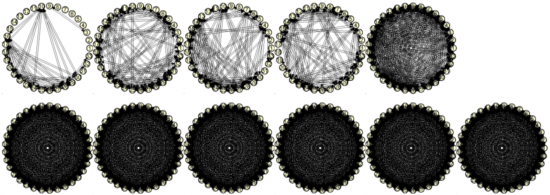
ED induced by TMS, SL on 1s windows,

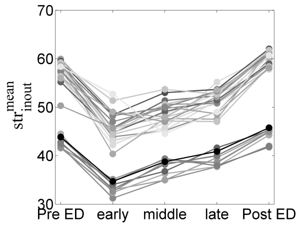


One episode



29 episodes Average strength

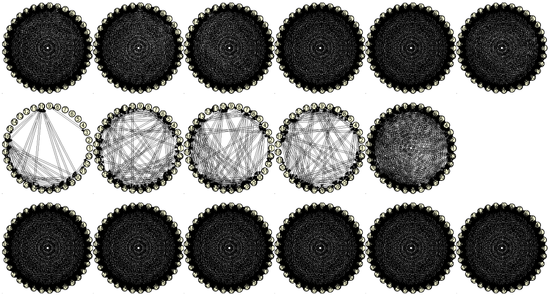
SL iCOH(α) SL (other parameters)



subject 1 with GGE

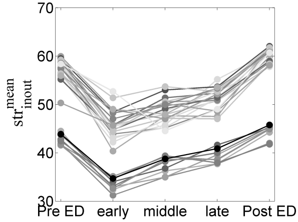
ED induced by TMS, SL on 1s windows,

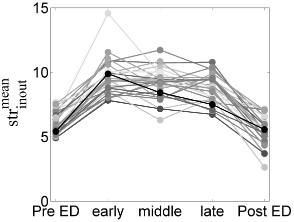
One episode



29 episodes   Average strength

SL   iCOH(α)   SL (other parameters)

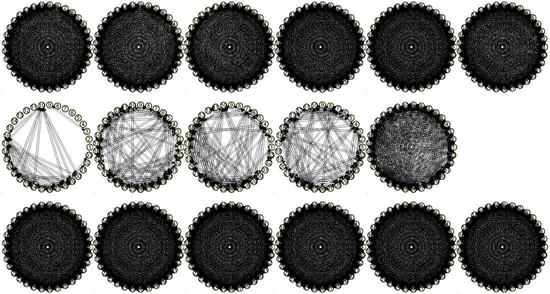




subject 1 with GGE

ED induced by TMS, SL on 1s windows,

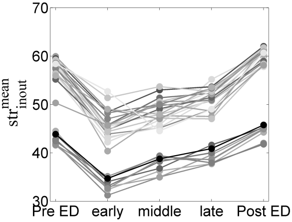
One episode



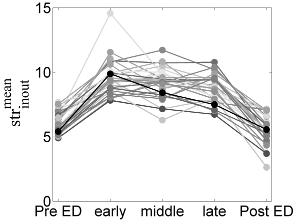
29 episodes

Average strength

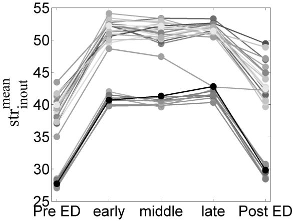
SL



iCOH(α)



SL (other parameters)



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- PMIME is model-free and almost parameter-free, can estimate nonlinear direct causal effects in the presence of many variables
- How can we learn the underlying dynamics of high-dimensional time series?

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collaborators in the research team

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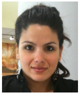
Christos Koutlis, PostDoc



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