A Class of Hamiltonian Systems with Fractional Power Nonlinearities

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Introduction

Questions:

- What is the dynamical behavior of a single Hamiltonian oscillator, whose potential is governed by fractional power nonlinearities?
- Can a generalization of the usual trigonometric functions $\sin, \cos$ be applied to treat this problem?
- What can we learn about such a Hamiltonian system of two or more such coupled oscillators?

Introduction-Simple Harmonic Oscillator

Let us start with the harmonic oscillator from Classical Mechanics

- Introduce the regular trigonometric functions
$$y_1(t) = \sin(\omega t), \quad y_2(t) = \cos(\omega t)$$
as solutions to the Cauchy problem of the linear harmonic oscillator:
$$y''(t) + \omega^2 y(t) = 0, \quad (y(0), y'(0)) = (y_{0_1}, y_{0_2}) \in \mathbb{R}, \omega \in \mathbb{R}^+$$
satisfied by the following expression:
$$y(t) = y_{0_1} \cos(\omega t) + (y_{0_2}/\omega) \sin(\omega t)$$

Simple Harmonic Oscillator

- Transformation to a system of first order ODEs yields:
$$y' = x, \quad x' = -\omega y$$
which then satisfies:
$$|x|^2 + |y|^2 = 1$$
- Substituting $x(t) = \sin(\omega t), \quad y(t) = \cos(\omega t)$:
$$|\sin(t)|^2 + |\cos(t)|^2 = 1$$
consequently leads to the Pythagorean trigonometric identity.

Generalizing the Trigonometric Functions

Hollomon's Nonlinear Oscillator

Hollomon, John Herbert. "Tensile deformation."

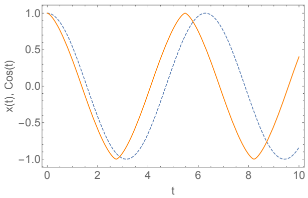
- Let us introduce the generalized trigonometric functions:
  $$y_1(t) = \sin_p(t), \quad y_2(t) = \cos_p(t)$$
  as solutions to the nonlinear Cauchy problem:
  $$\left[\phi^{-1}\left(y'\right)\right]' + \phi(y) = 0, \quad y(0) = 0, y'(0) = 1$$
  where $\phi : \Omega_1 \subset \mathbb{R} \to \Omega_2 \subset \mathbb{R}$, Ω_1, Ω_2 with:
  $$\phi(\epsilon) = |\epsilon|^{p-2}\epsilon, \quad p \in \mathbb{R}^+\{0\}$$
  a power law function.

Generalizing the Trigonometric Functions

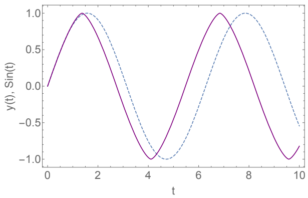
Dynamical System of Two ODEs

- Transformation to a system of first order ODEs yields:
$$x' = -\phi(y) = -|y|^{p-2}y, \quad y' = \phi(x) = |x|^{p-2}x$$
which then can be proved it satisfies:
$$|x|^p + |y|^p = 1$$
- Substituting $x(t) = \sin_p(t), \quad y(t) = \cos_p(t)$:
$$|\sin_p(t)|^p + |\cos_p(t)|^p = 1$$
consequently leads to the Generalized Pythagorean identity.
- For $p = 2$ the system drops back to the linear case and
$$\sin_{p=2}(t) = \sin(t), \cos_{p=2}(t) = \cos(t) \quad .$$

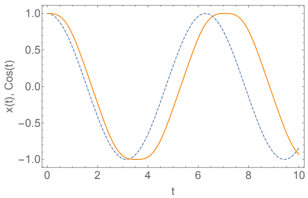
Plotting $\sin_p(t), \cos_p(t)$



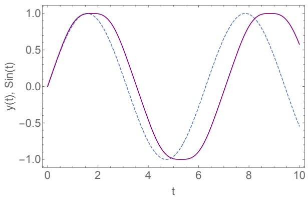
(a) $x(t)$ for $p = 1.5$.



(b) $y(t)$ for $p = 1.5$.

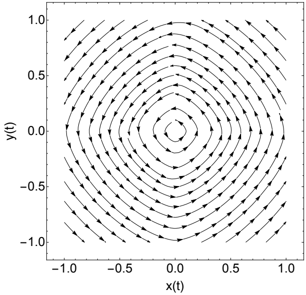


(c) $x(t)$ for $p = 3$.

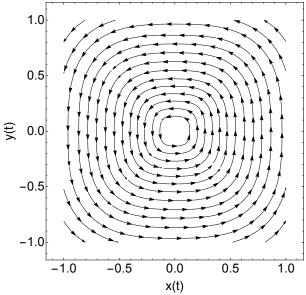


(d) $y(t)$ for $p = 3$.

Phase Portraits



(e) $p = 1.5$



(f) $p = 3$

Figure: Plots of the phase portraits for various p values and initial conditions $x(0) \in [-1, 1], y(0) \in [-1, 1]$.

Generalized Sine

Inverse Function

- The inverse generalized trigonometric $\sin_p(t)$ is defined as:

$$\sin_p^{-1}(t) = \begin{cases} \int_0^t \frac{ds}{\sqrt[p]{(1-s^p)^{p-1}}}, & 0 \leq t \leq 1 \\ -\int_0^{-t} \frac{ds}{\sqrt[p]{(1-s^p)^{p-1}}}, & -1 \leq t \leq 0 \end{cases}$$

  has two branches, is continuous and is defined over the domain $t \in [-1, 1]$ for $p \in \mathbb{R}^+$.
- However, an explicit expression for $\sin_p(t)$ has not been yet acquired.

Generalized Sine

Using Hypergeometric functions

- We can prove that it holds:

$$\sin_p^{-1}(t) = t_2F_1\left(1-\tfrac{1}{p},\tfrac{1}{p},1+\tfrac{1}{p};t^p\right), 0 \leq t \leq 1$$

  where:

$${}_2F_1\left(\alpha,\beta,\gamma;z\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}$$

  is the ordinary Hypergeometric Gauss function.

- The relation is consistent with the expected results, since:

$$\lim_{p\to 2}\sin_p^{-1}(t) = \lim_{p\to 2} t_2F_1\left(1-\frac{1}{p},\frac{1}{p},1+\frac{1}{p};t^p\right) = t_2F_1\left(1/2,2,3/2;t^2\right)$$

  and

$$t_2F_1\left(1/2,2,3/2;t^2\right) = \sin^{-1}(t)$$

Generalized Sine

Implicit Relation

- We can now derive the following sine function in an implicit way:

$$\sin_p(t) = \begin{cases} \dfrac{t}{{}_2F_1\left(1-\frac{1}{p},\frac{1}{p},1+\frac{1}{p};\left(\frac{t}{{}_2F_1\left(1-\frac{1}{p},\frac{1}{p},1+\frac{1}{p};(\cdots)^p\right)}\right)^p\right)} & 0 \leq t \leq 1 \\ -\dfrac{t}{{}_2F_1\left(1-\frac{1}{p},\frac{1}{p},1+\frac{1}{p};\left(-\frac{t}{{}_2F_1\left(1-\frac{1}{p},\frac{1}{p},1+\frac{1}{p};(\cdots)^p\right)}\right)^p\right)}, & -1 \leq t \leq 0 \end{cases} \tag{1}$$

- The generalized cosine function is then given by:

$$\cos_p(t) = \begin{cases} \sqrt[p]{1-|\sin_p(t)|^p}, & 0 \leq t \leq 1 \\ -\sqrt[p]{1-|\sin_p(t)|^p}, & -1 \leq t \leq 0 \end{cases} \tag{2}$$

Plot of $\sin_p(t)$

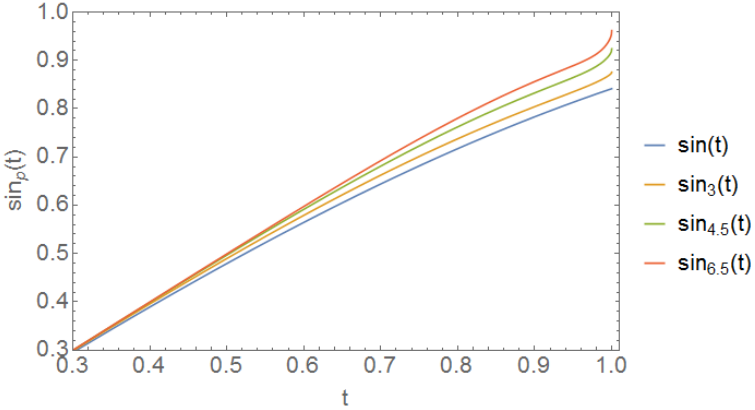


Figure: Plots of $\sin_p(t)$ for various values of $p = 2, p = 3, p = 4.5, p = 6.5$ and $t \in [0, 1]$.

Application to One Degree of Freedom

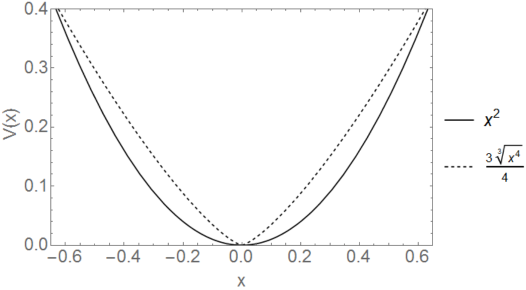
Nonlinear Spring-Mass System

- A direct application of these generalized sine functions is the nonlinear spring-mass system under the Hollomon law:
$$F(x, \dot{x}, t) = -k|x|^{p-2}x, \quad 1 < p \leq 2, k > 0$$
with an equation of motion:
$$m\ddot{x} + k|x|^{p-2}x = 0$$
- The potential function is given by:
$$V(x; p) = \frac{k|x|^p}{p}$$
- Nonlinear spring-mass system is solved by the generalized trigonometric functions:
$$(y_1(t), y_2(t)) = (\sin_p(t), \cos_p(t))$$

Application to One Degree of Freedom

The case of $p = 4/3$

- We choose $k/m = 1$ and $p = 4/3$ to get:
  $$\ddot{x} + x^{1/3} = 0$$
  The corresponding potential is given by:
  $$V(x) = (3/4)x^{4/3}$$



Application to One Degree of Freedom

Analytical Solution

A first integral of the motion can be derived by the ODE by direct integration:

$$\dot{x}^2 + (3/2)(\pm x)^{4/3} = c, \quad x \geq 0, x < 0$$

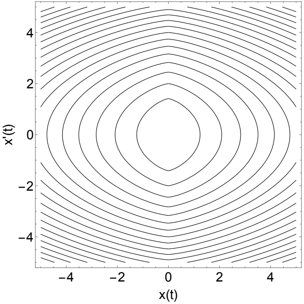


Figure: Phase portrait of the two first integrals of motion for $x \in [-5, 5]$, $\dot{x} \in [-5, 5]$ and $c \in [2, 40]$.

Application to One Degree of Freedom

Analytical Solution

- An implicit analytical solution can be provided involving elliptic integrals:
$$2\sqrt{6c}\left[K_2 - K_1\right]^2 = t^2, \quad c \in \mathbb{R}$$
- The elliptic functions K_1, K_2 depend on the $\arcsin(z)$:
$$K_1 = F\left[\arcsin\left[\left(\frac{3}{2c}\right)^{1/4} x^{1/3}\right]; -1\right], K_2 = E\left[\arcsin\left[\left(\frac{3}{2c}\right)^{1/4} x^{1/3}\right]; -1\right]$$
- The turning point is given by:
$$x_{tp} = (2/3)^{3/4} c^{3/4}$$
- The period of the solution is:
$$P = 4\sqrt{2}(6c)^{1/4}\left(E(-1) - F(-1)\right), \quad c \in \mathbb{R}$$

Application to One Degree of Freedom

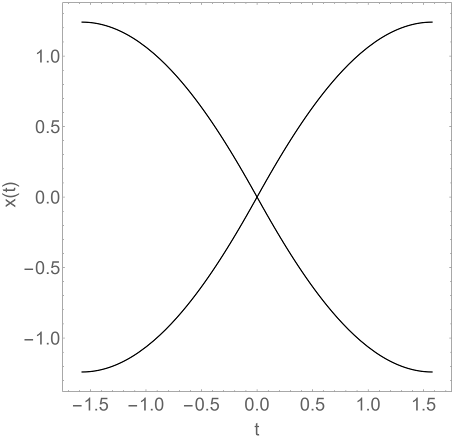


Figure: Parametric plot of the analytical solution for $t \in [-1, 1]$ and $c \to 2$ corresponding to the innermost curve in the phase space plot above.

Application to One Degree of Freedom

Fourier Series Approximation

- Those analytical expressions are useful, but their drawback is their domain $t \in [-1, 1]$. We suggest Fourier expansion to acquire a better approximation of the periodic solutions.
- We consider that the symbolic solution $x(t)$ can be approximated by:
$$x(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\{A_n \cos(n\omega t) + B_n \sin(n\omega t)\}$$
- Direct application to the ODE leads to the following trigonometric equation:
$$\sum_{n=1}^{\infty} A_{2n-1} \cos\left((2n-1)\omega t\right) = \left[\sum_{n=1}^{\infty} A_{2n-1} \left((2n-1)\omega\right)^2 \cos\left((2n-1)\omega t\right)\right]^3$$

Application to One Degree of Freedom

Fourier Series Approximation

- We begin by expanding and equating the first two terms:
$$A_1\cos(\omega t) + A_3\cos(3\omega t) = \left[A_1\omega^2\cos(\omega t) + 9\omega^2 A_3\cos(3\omega t)\right]^3$$
- Consistent with the left side of the original equation, only terms up to $\cos(3\omega t)$ should be retained. This leads to two equations involving terms proportional to $\cos(\omega t)$ and $\cos(3\omega t)$ respectively:

$$\begin{cases} A_1^2 + 9A_1A_3 + 162A_3^2 = \frac{4}{3\omega^6} \\ A_1^3 + 54A_3A_1^2 + 2187A_3^3 = \frac{4A_3}{\omega^6} \end{cases}$$

Application to One Degree of Freedom

Fourier Series Approximation

- Insight can be gained by numerical evaluation of A_1, A_3.

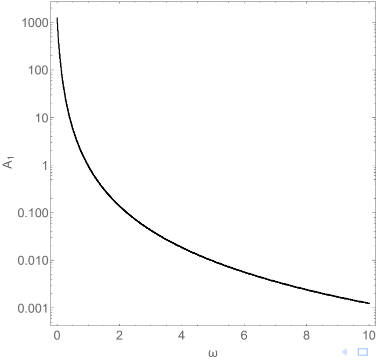
Table: Numerical values of the coefficients A_1, A_3, A_5 for $\omega \approx 0.996167$.

A_1	A_3	A_5
-1.25523	0.027139	-0.00384421
1.25523	-0.027139	0.00384421

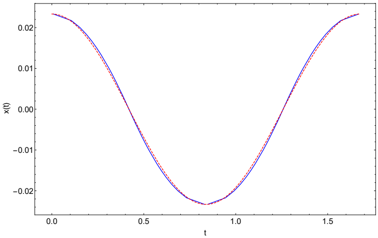
Application to One Degree of Freedom

Fourier Series Approximation

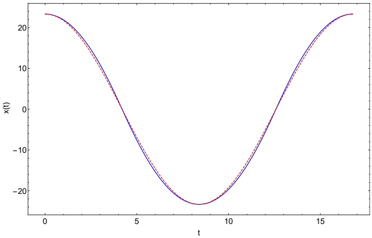
- Although the values of A_1 vary rapidly with ω, the ratios A_3/A_1 and A_5/A_1 are essentially constant at $A_3/A_1 = -0.0216208$ and $A_5/A_1 = 0.0030625543$ except for small values of $\omega < 0.01$.



Comparing Single-Cosine with the Symbolic Expression



(a) $c = 0.01$



(b) $c = 100$

Agreement is excellent for c in the range of 0.01 to 100.

Analysis on Coupled Oscillators

- We consider the following system of nonlinear coupled oscillators:

$$\ddot{x} = -|x|^{n-1}x + k|y-x|^{n-1}(y-x)$$
$$\ddot{y} = -|y|^{n-1}y - k|y-x|^{n-1}(y-x)$$

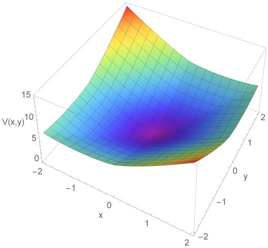
- The corresponding potential reads:

$$V(x,y) = \frac{1}{(n+1)}x^{n+1} + \frac{k}{(n+1)}(y-x)^{n+1} + \frac{1}{(n+1)}y^{n+1}$$

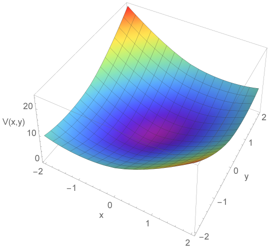
- and the Hamiltonian is:

$$H(x,y) = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{(n+1)}\left(x^{n+1} + k(y-x)^{n+1} + y^{n+1}\right)$$

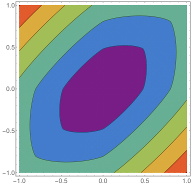
Plot of the Potential

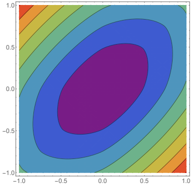


(c) $n = 1/3$.



(d) $n = 3/5$.





Application to Two Degrees of Freedom

The case of $n = 1/3$

- The potential reads:
$$V(x, y) = (3/4)\left(x^{4/3} + k(y-x)^{4/3} + y^{4/3}\right)$$
- System's Energy E coincides with the Hamiltonian:
$$E = (1/2)\left(\dot{x}^2 + \dot{y}^2\right) + (3/4)\left(x^{4/3} + k(y-x)^{4/3} + y^{4/3}\right)$$
and remains invariant under the scaling transformation:
$$x = \lambda\hat{x}, \quad t = \lambda^{1/3}\tau$$

Application to Two Degrees of Freedom

Special Periodic Solutions

- Plugging in the system of ODEs $\hat{x} = \hat{y}$, reduces to the problem of one oscillator, solved previously.
$$\ddot{\hat{x}} = -\hat{x}^{1/3} \qquad \textit{(In phase solution)}$$
- Setting in the system of ODEs $\hat{x} = -\hat{y}$, leads to the following problem of one degree of freedom:
$$\ddot{\hat{x}} = -(2^{1/3}k + 1)\hat{x}^{1/3} \qquad \textit{(Out of phase solution)}$$
which is again solved by the theory of the single oscillator discussed earlier.

Poincaré Sections of the Coupled HOS

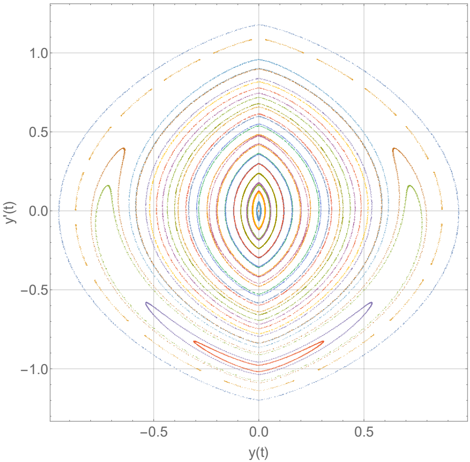


Figure: $k = 0.01$

Poincaré Sections of the Coupled HOS

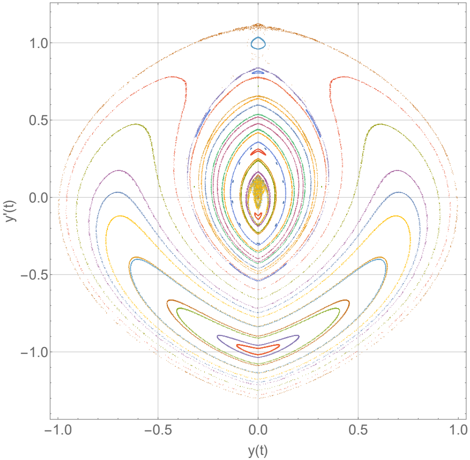


Figure: $k = 0.1$

Poincaré Sections of the Coupled HOS

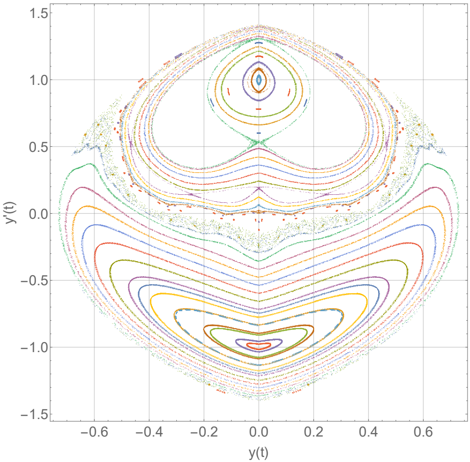


Figure: $k = 1$

Conclusion & Future Work

Remarks

- Generalized trigonometric functions $\sin_p(t), \cos_p(t)$ constitute solutions to the nonlinear harmonic oscillator, since they satisfy the generalized Pythagorean identity:
$$|\sin_p(t)|^p + |\cos_p(t)|^p = 1$$
- We present an analytical solution and a Fourier approximate for the nonlinear spring-mass system under Hollomon's law.
- Poincar$\acute{e}$ sections for the coupled HOS show both regions of stability and weak chaos for various values of k.

Conclusion & Future Work

Future Work

- Extension of our study to the case of $n = 3/5$.
- Elaborate on our study of the class of periodic solutions called Simple Periodic Orbits (SPOs).
- Search for Fourier approximates in the case of coupled HOS.
- What about more degrees of freedom?

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