Integrable and Non-Integrable Systems of Competing Populations

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Consider a Lotka-Volterra system of ODEs

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}_i = \sum_{j=1}^{k} a_{ij} x_i x_j$$

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The goal is to study the integrability and solvablity of the system by imposing the strong Panleve property, that is we look for solutions which have **only (simple) poles** as their movable singularities. Moreover, their Laurent series expansions around them depend on n free parameters (including the singularity itself).

Case $n = 3$ and $k = 3$

In this case the system is

$$\begin{aligned}
\dot{x}_1 &= x_1(Ax_2 + Bx_3)\\
\dot{x}_2 &= x_2(-Ax_1 + Cx_3)\\
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Let t_* denote the location of the movable singularity and $\tau = t - t_*$. Consider the Laurent series expansions of the solutions near t_*.

$$\begin{aligned} x_1 &= \alpha\tau^p + \dots \\ x_2 &= \beta\tau^q + \dots \\ x_3 &= \gamma\tau^s + \dots \end{aligned}$$

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where $p, q \in \mathbb{Z}$.
Substitute these expressions into the system and we consider the most singular case

$$p = -1, \quad q = -1, \quad s = -1$$

Thus we are reduced to the situation

$$\begin{aligned} x_1 &= \alpha\tau^{-1} + a_0 + a_1\tau + \ldots + a_{r-1}\tau^{r-1} + \ldots \\ x_2 &= \beta\tau^{-1} + b_0 + b_1\tau + \ldots + b_{r-1}\tau^{r-1} + \ldots \\ x_3 &= \gamma\tau^{-1} + c_0 + c_1\tau + \ldots + c_{r-1}\tau^{r-1} + \ldots \end{aligned}$$

Substitute these expressions into the system and obtain the following condition on the coefficients α, β and γ.

$$\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & A & B \\ -A & 0 & C \\ -B & -C & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

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$$0 = \det \begin{pmatrix} -1 & A & B \\ -1 & 0 & C \\ -1 & -C & 0 \end{pmatrix} = \det \begin{pmatrix} 0 & -1 & B \\ -A & -1 & C \\ -B & -1 & 0 \end{pmatrix} = \det \begin{pmatrix} 0 & A & -1 \\ -A & 0 & -1 \\ -B & -C & -1 \end{pmatrix}$$

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This leads to the condition $C = B - A$. Moreover, the solution of this system is

$$\alpha = \text{free}, \quad \beta = \frac{1 - B\alpha}{C}, \quad \gamma = \frac{A\alpha - 1}{C}$$

Substitute $C = B - A$ and $x_3 = h - x_1 - x_2$ into the system

$$\begin{gathered} \dot{x_1} = x_1(Bh + (A - B)x_2 - Bx_1) \\ \dot{x_2} = x_2((B - A)h - Bx_1 + (A - B)x_2) \\ \dot{x_3} = x_3(-Bx_1 - (B - A)x_2) \end{gathered}$$

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\end{gathered}$$

Therefore, we obtain

$$\frac{\dot{x_1}}{x_1} - \frac{\dot{x_3}}{x_3} = Bh, \quad \frac{\dot{x_2}}{x_2} - \frac{\dot{x_3}}{x_3} = (B - A)h$$

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giving us

$$\frac{x_1}{x_3}e^{-Bht} = K_1, \quad \frac{x_2}{x_3}e^{-(B-a)ht} = K_2$$

where K_1, K_2 are free constants.

Hence,

$$\frac{\dot{x_3}}{x_3} = -Bx_1 - (B - A)x_2 = -BK_1 x_3 e^{Bht} - (B - A)x_3 K_2 e^{(B-a)ht}$$

$$\frac{\dot{x_3}}{x_3^2} = -BK_1 e^{Bht} - (B - A)K_2 e^{(B-a)ht}$$

$$\frac{1}{x_3} = K_1 e^{Bht} + K_2(B - A)e^{(B-a)ht} + K_3$$

Finally, the integral is

$$x_1 = \frac{K_1 e^{Bht}}{K_1 e^{Bht} + K_2(B - A)e^{(B-a)ht} + K_3},$$

$$x_2 = \frac{K_2 e^{(B-A)ht}}{K_1 e^{Bht} + K_2(B - A)e^{(B-a)ht} + K_3},$$

$$x_3 = \frac{1}{K_1 e^{Bht} + K_2(B - A)e^{(B-a)ht} + K_3}$$

It can be observed in the study of particular cases that the coefficients of the antisymmetric matrix A satisfy the following restraining relations

$$a_{ij} = a_{i,k} + a_{k,j}$$

for all $1 \leq i < k < j \leq n$.

To obtain the general solution, we rewrite the equations, using that $H(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} x_i = h =$ constant, as follows

$$\dot{x_1} = x_1(a_{12}x_2 + a_{13}x_3 + \ldots + a_{1,n-1}x_{n-1} + a_{1n}(h - x_1 - x_2 - \ldots x_{n-1}))$$

$$\dot{x_2} = x_2(-a_{12}x_1 + a_{23}x_3 + \ldots + a_{2,n-1}x_{n-1} + a_{2n}(h - x_1 - x_2 - \ldots x_{n-1}))$$

$$\ldots$$

$$\dot{x_{n-1}} = x_{n-1}(-a_{1,n-1}x_1 + a_{2,n-1}x_2 - \ldots - a_{n-2,n-1}x_{n-2} + a_{n-1,n}(h - x_1 - x_2 - \ldots x_{n-1})$$

$$\dot{x_n} = x_n(-a_{1,n}x_1 - a_{2,n}x_2 - \ldots - a_{n-2,n}x_{n-2} - a_{n-1,n}x_{n-1})$$

Simplify

$$\dot{x_1} = x_1(a_{1n}h + (a_{12} - a_{1n})x_2 + (a_{13} - a_{1n})x_3 + \ldots + (a_{1,n-1} - a_{1n})x_{n-1} - a_{1n}x_1)$$

$$\dot{x_2} = x_2(a_{2n}h - (a_{12} + a_{2n})x_1 + (a_{23} - a_{2n})x_3 + \ldots + (a_{2,n-1} - a_{2n})x_{n-1} - a_{2n}x_2)$$

$$\ldots$$

$$\dot{x_{n-1}} = x_{n-1}(a_{n-1,n}h - (a_{1,n-1} + a_{n-1,n})x_1 - (a_{2,n-1} + a_{n-1,n})x_2 - \ldots$$
$$-(a_{n-2,n-1} + a_{n-1,n})x_{n-2} - a_{n-1,n}x_{n-1}))$$

$$\dot{x_n} = x_n(-a_{1,n}x_1 + a_{2,n}x_2 + \ldots - a_{n-2,n}x_{n-2} + a_{n-1,n}x_{n-1})$$

Then using the restraining relations we obtain

$$\dot{x_1} = x_1(a_{1n}h - a_{2n}x_2 - a_{3n}x_3 + \ldots - a_{n-1,n}x_{n-1} - a_{1n}x_1)$$

$$\dot{x_2} = x_2(a_{2n}h - a_{1n}x_1 - a_{3n}x_3 - \ldots - a_{n-1,n}x_{n-1} - a_{2n}x_2)$$

$$\ldots$$

$$\dot{x_{n-1}} = x_{n-1}(a_{n-1,n}h - a_{1,n}x_1 - a_{2,n}x_2 - \ldots - a_{n-2,n}x_{n-2} - a_{n-1,n}x_{n-1})$$

$$\dot{x_n} = x_n(-a_{1,n}x_1 + a_{2,n}x_2 + \ldots - a_{n-2,n}x_{n-2} + a_{n-1,n}x_{n-1})$$

Observe

$$\dot{x_1} = x_1(a_{1n}h - a_{2n}x_2 - a_{3n}x_3 + \ldots - a_{n-1,n}x_{n-1} - a_{1n}x_1)$$

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$$\dot{x_{n-1}} = x_{n-1}(a_{n-1,n}h - a_{1,n}x_1 - a_{2,n}x_2 - \ldots - a_{n-2,n}x_{n-2} - a_{n-1,n}x_{n-1})$$

$$\dot{x_n} = x_n(-a_{1,n}x_1 + a_{2,n}x_2 + \ldots - a_{n-2,n}x_{n-2} + a_{n-1,n}x_{n-1})$$

Denote by $Q = -a_{1n}x_1 - a_{2n}x_2 - a_{3n}x_3 + \ldots - a_{n-1,n}x_{n-1}$.

$$Q = -a_{1n}x_1 - a_{2n}x_2 - a_{3n}x_3 + \ldots - a_{n-1,n}x_{n-1}$$

Then we have

$$\frac{\dot{x_1}}{x_1} = a_{1n}h + Q = a_{1n}h + \frac{\dot{x_n}}{x_n}$$

$$\frac{\dot{x_2}}{x_2} = a_{2n}h + Q = a_{2n}h + \frac{\dot{x_n}}{x_n}$$

$$\ldots$$

$$\frac{\dot{x_{n-1}}}{x_{n-1}} = a_{n-1,n}h + Q = a_{n-1,n}h + \frac{\dot{x_n}}{x_n}$$

$$\frac{\dot{x_n}}{x_n} = Q$$

Finally, we have the solution

$$x_i(t) = C_i x_n(t) e^{a_{in} ht}$$

for all $i = 1, \ldots, n-1$. To find $x_n(t)$,

$$\frac{\dot{x_n}}{x_n} = -a_{1n}x_1 - a_{2n}x_2 - a_{3n}x_3 + \ldots - a_{n-1,n}x_{n-1} = -x_n \sum_{i=1}^{n-1} C_i a_{ih} e^{a_{in} ht}$$

$$\frac{\dot{x_n}}{x_n^2} = -\sum_{i=1}^{n-1} C_i a_{in} e^{a_{in} ht}$$

$$\frac{1}{x_n} = \sum_{i=1}^{n-1} C_i a_{ih} \int_0^t e^{a_{in} ht} \mathrm{d}t$$

Therefore, we have

$$x_n = \frac{h}{\sum_{i=1}^{n-1} C_i e^{a_{in} ht} + C_n}$$

Consider the following perturbation of the original system

$$\dot{x_1} = \lambda_1 x_1 + x_1(a_{12}x_2 + a_{13}x_3 + \ldots + a_{1,n-1}x_{n-1} + a_{1n}x_n)$$

$$\dot{x_2} = \lambda_2 x_2 + x_2(-a_{12}x_1 + a_{23}x_3 + \ldots + a_{2,n-1}x_{n-1} + a_{2n}x_n)$$

$$\ldots$$

$$\dot{x_{n-1}} = \lambda_{n-1}x_{n-1} + x_{n-1}(-a_{1,n-1}x_1 + a_{2,n-1}x_2 - \ldots - a_{n-2,n-1}x_{n-2} + a_{n-1,n}x_n)$$

$$\dot{x_n} = \lambda_n x_n + x_n(-a_{1,n}x_1 - a_{2,n}x_2 - \ldots - a_{n-2,n}x_{n-2} - a_{n-1,n}x_{n-1})$$

The method that we used above cannot be applied to this system since the Hamiltonian $H(x_1, x_2, ..., x_n) = \sum_{i=1}^{k} x_i$ cannot be constant. Moreover, this system turns out to be non-integrable.

Reference: T. Bountis and P. Vanheacke, Lotka-Volterra systems satisfying a strong Painleve property, Physics Letters A, Volume 380, Issue 47, 9 December 2016, pp 3977-3982

THANK YOU!