Hyperbolic Theory of Dynamical Systems

A short introduction with a view toward examples

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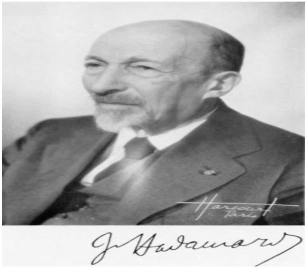
Outline

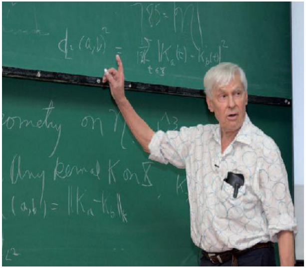
The giants

Jacques S Hadamard (1865-1963)



Henri Poincaré (1854-1912)

Dmitri V Anosov (1936-2014)



Stephen Smale (1930...)

Milestones of Hyperbolic Theory

A rather incomplete list

- **Global Stability of Dynamical Systems**
  Hartman–Grobman Theorem, Stable Manifold Theorem, Closing Lemma, Shadowing, Genericity, Ω–Stability Theorem, Structural Stability Theorem
- **Geometry and Dynamics**
  hyperbolic actions, hyperbolic geometry, geodesic flow, Anosov diffeomorphisms and flows, billiards, recurrence
- **Construction of guiding examples**
  conditions for hyperbolicity, Shilnikov's theorem, calculation of invariant sets, verification of hyperbolicity in concrete examples, asymptotic behavior of the system
- **Ergodic Theory**
  equidistribution and recurrences, SRB-measures, zeta–functions, topological entropy

The linear case

Definition

Let $A \in M(n \times n, \mathbb{R})$. We shall call A hyperbolic if no eigenvalue of A has norm equal to 1. The linear map corresponding to A, that is, $T : \mathbb{R}^n \to \mathbb{R}^n,\ T(x) = Ax$ is also called hyperbolic.

Example

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ with eigenvalues $\lambda_1 = \frac{3+\sqrt{5}}{2} > 1,\ \lambda_2 = \lambda_1^{-1} < 1$ and eigenvectors $e_1 = (\frac{1+\sqrt{5}}{2}, 1)$ and $e_2 = (\frac{1-\sqrt{5}}{2}, 1)$.

- e_1, e_2 form a basis of $\mathbb{R}^2$ or $\mathbb{R}^2 = \langle e_1 \rangle \oplus \langle e_2 \rangle$
- For $T : \mathbb{R}^2 \to \mathbb{R}^2,\ T(x, y) = (2x + y, x + y)$ we have that $T\ |_{\langle e_1 \rangle}$ is a dilation and $T\ |_{\langle e_2 \rangle}$ a contraction.

The nonlinear case

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}^n$, of class C^r, $r \geq 1$, and Λ a smooth subset of $\mathbb{R}^n$, invariant under f (i.e. $f(\Lambda) = \Lambda$). We say that Λ is hyperbolic for f if:

- $T_x\mathbb{R}^n = E^s(x) \oplus E^u(x), \ \forall x \in \Lambda.$
- $d_x f(E^s(x)) \subseteq E^s(f(x))$ and $d_x f(E^u(x)) \subseteq E^u(f(x))$
- $\| d_x f^n|_{E^s(x)} \|, \ \| d_x f^{-n}|_{E^u(x)} \| \leq C\tau^n, \ C > 0, \ 0 < \tau < 1, \ n \in \mathbb{N}.$

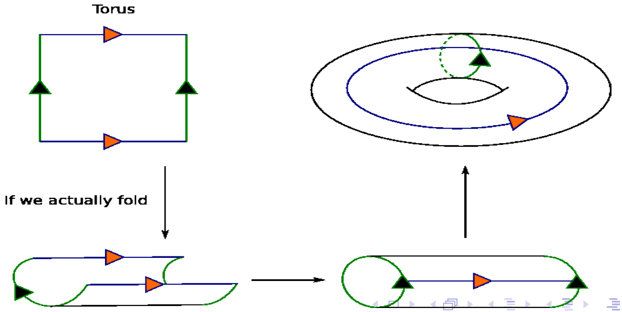
Example

For $f : \mathbb{R}^2 \to \mathbb{R}^2, \ f(x, y) = (2x + y^2, x^3 + \frac{1}{3}y)$, the origin is a (trivial) hyperbolic set. We can study the behavior of f in a neighborhood of it using the Hartman-Grobman theorem.

The 2–dimensional torus

Or, how to "construct" a pretzel

- If $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, define $(x_1, y_1) \sim (x_2, y_2)$ iff $x_1 - x_2 \in \mathbb{Z}$ and $y_1 - y_2 \in \mathbb{Z}$.
- The set of all points of $\mathbb{R}^2$ equivallent to (x, y) will be denoted as $[(x, y)]$.
- The set of all equivallence classes will be denoted as $\mathbb{R}^2/\mathbb{Z}^2$.
- $\mathbb{R}^2/\mathbb{Z}^2 \simeq \mathbb{T}^2$.



A hyperbolic toral automorphim

- We return to $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (2x + y, x + y)$.
- It's easy to show that if $(x_1, y_1) \sim (x_2, y_2)$ then $T(x_1, y_1) \sim T(x_2, y_2)$.
- We have thus defined a mapping $L : \mathbb{T}^2 \to \mathbb{T}^2$.

Theorem

$\mathbb{T}^2$ is a hyperbolic set for L. Periodic orbits of L are dense in $\mathbb{T}^2$, L has a dence orbit which is not periodic and $P_n(F_L) = \lambda_1^n + \lambda_1^{-n} - 2$.

Proof.

Points with rational coordinates, and only these, are periodic orbits of L. Actually one can find a special "relation" between L and the Bernoulli shift. □

- Note that arbitrarily close points may have quite different future!
- One can easily generalize the example above in any dimension.

“Simple” chaos

Let us here recall the definition of chaos.

Definition

Let $f : M \to M$ and $N \subseteq M$ a compact subset, invariant under f. We say that f presents chaotic behavior in N if:

- The set of periodic orbits of f is dense in N.
- N contains a dense, non–periodic, orbit of f.

Since $\mathbb{T}^2$ is compact, we conclude that:

Theorem

Mapping $L : \mathbb{T}^2 \to \mathbb{T}^2$ *defined above is chaotic.*

Proof.

Actually, this is a corollary of the previous theorem. □

Things to remember

- Hyperbolic theory provides us with simple examples of chaotic behavior, which are amenable to analytical study.

Anything else?

Stability

When do two dynamical systems "have the same behavior"?

Definition

Let $f, g : X \to X$ be two continuous mappings of the topological space X. We shall call them topologically conjugate if a homeomorphism $h : X \to X$ exists, such that $g \circ h = h \circ f$.

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h\downarrow & & \downarrow h \\ X & \xrightarrow[g]{} & X \end{array}$$

Theorem

Every C^1 diffeomorphism $g : \mathbb{T}^2 \to \mathbb{T}^2$ which is sufficiently close to $L : \mathbb{T}^2 \to \mathbb{T}^2$, in the C^1–topology, is topologically conjugate to L.

Proof.

- We have to solve equation $g \circ h = h \circ L$, with respect to $h \in Hom(\mathbb{T}^2)$.
- Passing to the lift of $\mathbb{T}^2$, that is $\mathbb{R}^2$, we rewrite last equation as $\tilde{g} \circ (Id + \tilde{h}) = \tilde{h} \circ L - L \circ \tilde{h}$.
- Define $\mathcal{L}(\tilde{h}) = \tilde{h} \circ L - L \circ \tilde{h}$. Last equation becomes $\tilde{g} \circ (Id + \tilde{h}) = \mathcal{L}(\tilde{h})$.
- Define $T(\tilde{h}) = \tilde{g} \circ (Id + \tilde{h})$. Last equation becomes $T(\tilde{h}) = L(\tilde{h}) \Rightarrow \tilde{h} = \mathcal{L}^{-1} T(\tilde{h})$.
- Prove that when g is C^1–close to L, operator $\mathcal{L}^{-1} T$ is a contraction. Thus, it has a fixed point.
- Prove that the fixed point is what you were looking for.

Things to remember

So:

- Hyperbolic theory provides us with simple examples of chaotic behavior, which are amenable to analytical study.
- It also proves that hyperbolic chaos is a stable property, thus we cannot "sweep it under the carpet".

Anything else?

Attractors

Simple, hyperbolic, strange, chaotic....

Definition

Let $f : X \to X$ be a continuous dynamical system of the topological space X. The subset $A \subset X$ is called an attractor for f if there exists an open set $V \supset A$ such that $f(V) \subset V$ and $A = \cap_{n \in \mathbb{N}} f^n(V)$.

- A hyperbolic attractor is an attractor which is also a hyperbolic set.
- A chaotic attractor is an attractor which also fulfills the definition of chaos.
- A strange attractor is an attractor which has a fractal structure.

Example

The origin is a (simple) attractor for $f : \mathbb{R}^2 \to \mathbb{R}^2, \; f(x, y) = (\frac{1}{2}x, \frac{1}{2}y)$.

The DA attractor

Derived from Anosov

Let us return to the hyperbolic toral automorphism $L : \mathbb{T}^2 \to \mathbb{T}^2$ studied above.

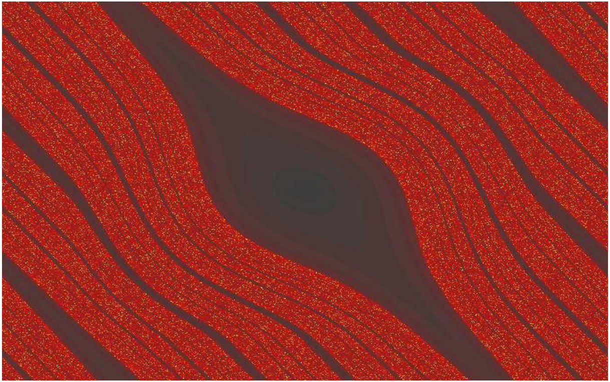
- Denote by p_0 the fixed point corresponding to the origin.
- Let $\delta : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $0 \leq \delta(x) \leq 1, \ \forall x \in \mathbb{R}$ and $\delta(x) = 0$ for $x \geq r_0$ and $\delta(x) = 1$ for $x \leq r_0/2$, where $r_0 > 0$ but "small enough".
- Consider the differential equations

$$\begin{aligned} \dot{u}_1 &= 0 \\ \dot{u}_2 &= u_2 \cdot \delta(\| (u_1, u_2) \|) \end{aligned}$$

  and denote by $\varphi^t(u_1, u_2)$ their flow.
- Define the "DA–diffeomorphism" $f : \mathbb{T}^2 \to \mathbb{T}^2$ as $f(u_1, u_2) = \varphi^\tau \circ L(u_1, u_2)$ where $\tau > 0$ any fixed number such that $e^\tau \lambda_2 > 1$.

Theorem

The DA–diffeomorphism has a non–wondering set consisting of the repelling fixed point p_0 and a set Λ, which is a hyperbolic chaotic attractor for f.



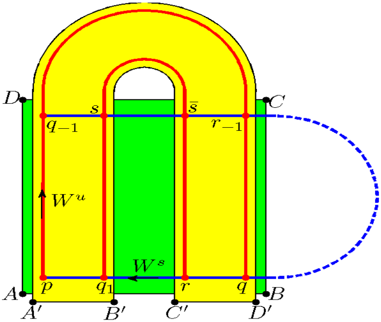
The DA attractor (in red).

Smale's horseshoe

(What really happened at the beaches of Rio)

Define a diffeomorpshism $\phi : \mathbb{R}^2 \cup \{\infty\} \to \mathbb{R}^2 \cup \{\infty\}$ as follows:

- The point at infinity (denoted by $\{\infty\}$) is repelling.
- There exists a square S on which ϕ acts as shown on the figure bellow, by expanding vertical directions and contracting the horizontal ones.



Definition

We define $H = \cap_{n\in\mathbb{Z}}\phi^n(S)$ and the mapping $\phi|_H : H \to H$. This mapping is called "the classical Smale's horseshoe".

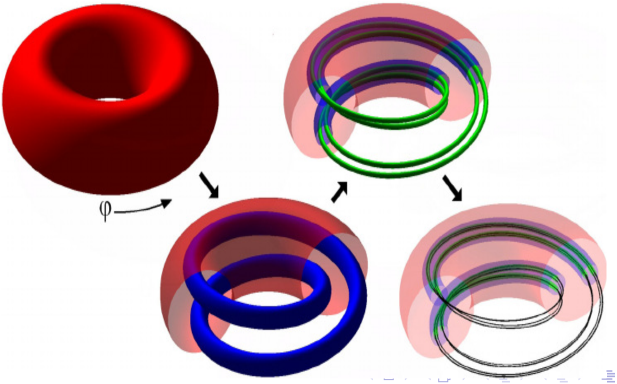
Theorem

Smale's horseshoe is a hyperbolic chaotic set.

The solenoid attractor
Or Smale–Williams attractor

- Consider the "solid torus" $\mathbb{S}^1 \times \mathbb{D}^2$, equipped with coordinates $\phi \in \mathbb{S}^1$, $(x, y) \in \mathbb{D}^2$.
- Define:

$$f : \mathbb{S}^1 \times \mathbb{D}^2 \to \mathbb{S}^1 \times \mathbb{D}^2$$
$$f(\phi, x, y) = (2\phi, \tfrac{1}{10}x + \tfrac{1}{2}cos\phi, \tfrac{1}{10}y + \tfrac{1}{2}sin\phi).$$



- It is really easy to prove that f is $1-1$ and that $f(\mathbb{S}^1 \times \mathbb{D}^2) \subset \mathbb{S}^1 \times \mathbb{D}^2$.

Definition

The solenoid attractor is defined to be $\mathcal{S} = \cap_{n\in\mathbb{N}} f^n(\mathbb{S}^1 \times \mathbb{D}^2)$, equipped with the dynamics $f|_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$.

Theorem

$\mathcal{S}$ is a hyperbolic chaotic attractor for f.

Proof.

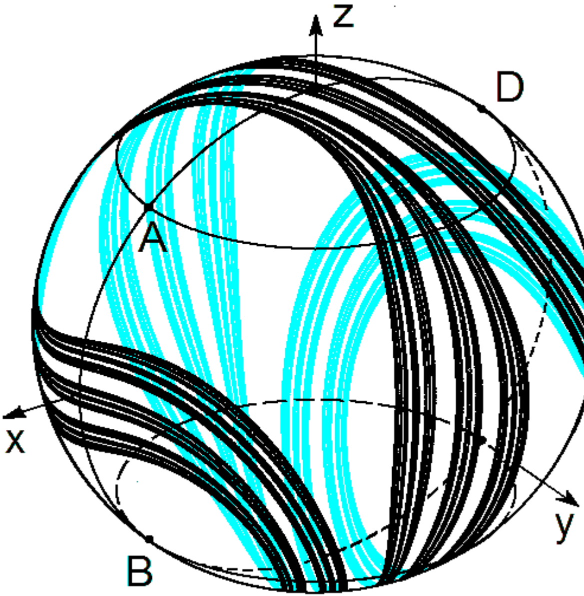
One easily verifies the so–called "cone conditions". □

The Plykin attractor

Consider the following differential equations on the $\mathbb{S}^2 \subset \mathbb{R}^3$.

- Flow down along circles of latidude:
$$\dot{x} = -\epsilon x y^2, \;\; \dot{y} = \epsilon x^2 y, \;\; \dot{z} = 0.$$
- Rotation around the z–axis:
$$\dot{x} = \pi(\tfrac{1}{\sqrt{2}} z + \tfrac{1}{2}) y, \;\; \dot{y} = -\pi(\tfrac{1}{\sqrt{2}} z + \tfrac{1}{2}) x, \;\; \dot{z} = 0.$$
- Flow down to the equator:
$$\dot{x} = 0, \;\; \dot{y} = \epsilon y z^2, \;\; \dot{z} = -\epsilon y^2 z.$$
- Rotation around the x–axis:
$$\dot{x} = 0, \;\; \dot{y} = -\pi(\tfrac{1}{\sqrt{2}} z + \tfrac{1}{2}) z, \;\; \dot{z} = \pi(\tfrac{1}{\sqrt{2}} z + \tfrac{1}{2}) y.$$

Following the flows of these o.d.e.'s for time equal to 1 unit, and then repeating this procedure, we have (for $\epsilon = 0.77$):


z
D
A
x
y
B

Things to remember

So:

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- It also proves that hyperbolic chaos is a stable property, thus we cannot “sweep it under the carpet”.
- It teaches us that “one can not escape chaos”, since chaotic sets are usually attracting. The chaotic attractors provided by the theory are also ammenable to analytical study (and really beautiful!).

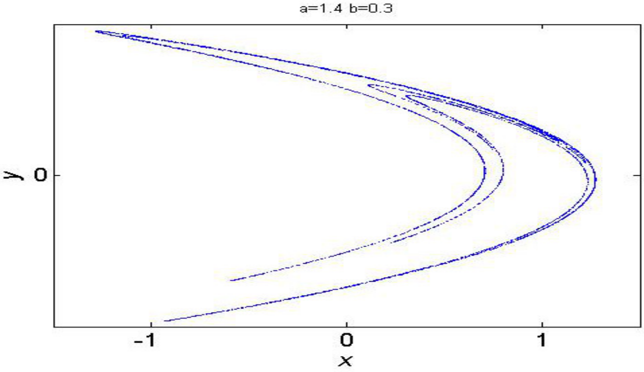
Anything else?

The map of Hénon has a horseshoe

Let us recall Hénon map:

$$h_s : \mathbb{R}^2 \to \mathbb{R}^2, \; h_s(x, y) = (1 + y - ax^2, bx),$$

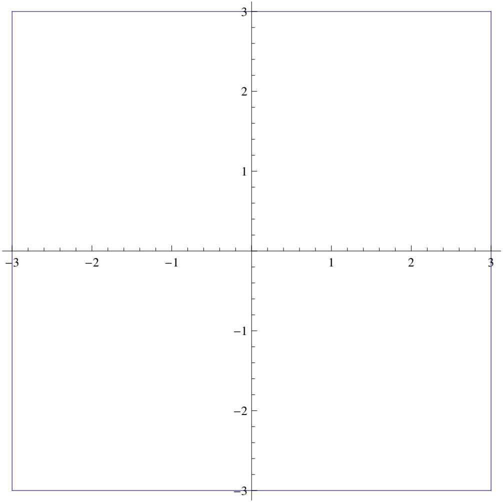
presented in: Hénon M, "A two–dimensional mapping with a strange attractor", Comm.Math.Phys., 50(1), 69-77,1976.



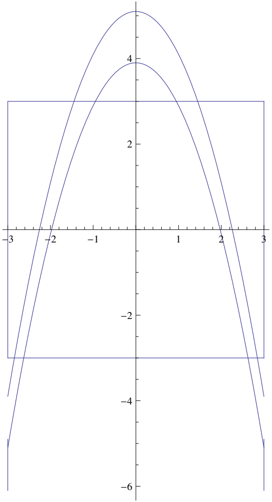
Nowadays its more common to study its topologically conjugate standard Hénon map:

$$h : \mathbb{R}^2 \to \mathbb{R}^2, \ h(x, y) = (y, -\beta x + a - y^2).$$

We consider the square $S = \{x, y \in \mathbb{R}^2, x, y \in [-3, 3]\}$.



Let us calculate the set $S \bigcap h(S)$.

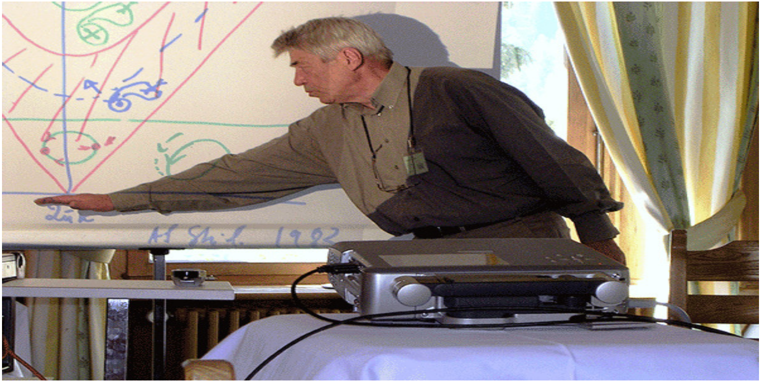


- Thus, the standard Hénon map possesses a geometrical horseshoe.
- Is it however "a true horseshoe", that is, a hyperbolic chaotic invariant set having the structure described by Smale?
- Yes, it is.
  Devaney R L, Nitecki Z,
  "Shift automorphisms in the Hénon mapping",
  Comm.Math.Phys., 67, 137-146, 1979.

Leonid Pavlovich Shilnikov, 1934-2011

Or how to prove that there is a horseshoe arround (and much much more...)

- Actually, since then, we have found horseshoes in many dynamical systems.
- To accomplish that, we usually employ Shilnikov's theorem.



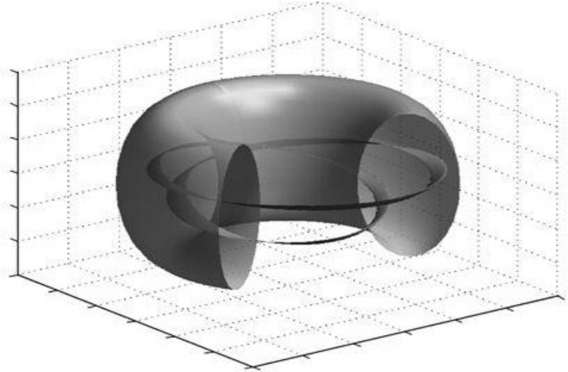
Alternately excited van del Pol oscillators

Let us consider the following system of differential equations:

$$\begin{aligned} \ddot{x} - (A\cos\frac{2\pi t}{T} - x^2)\dot{x} + \omega_0^2 x &= \epsilon y \cos(\omega_0 t) \\ \ddot{y} - (-A\cos\frac{2\pi t}{T} - y^2)\dot{y} + 4\omega_0^2 y &= \epsilon x^2 \end{aligned}$$

, where $T = \frac{2\pi N}{\omega_0} = 6, \ \omega_0 = 2\pi, \ A = 5, \ \epsilon = 0.5$.

- It's extended phase space is 5–dimensional.
- We can define a 4–dimensional Poincaré section $\mathcal{P}$.
- Poincaré map $T : \mathcal{P} \to \mathcal{P}$ has a forward invariant set $D \simeq \mathbb{S} \times \mathbb{D}^3$.
- Let us draw (the 3–dimensional projection of) D and $T(D)$.



"Example of a physical system with an attractor of the Smale–Williams type"
S P Kuznetsov
Phys.Rev.Lett. (95), 144101, 2005.

Kuznetsov gave us a detailed (numerical) study of:

- Construction of the Poincaré map.
- Existence of the forward invariant set D.
- Existence of a solenoid attractor.
- Numerical verification of its hyperbolicity (through cone conditions).

All these were made completely rigorous in:

"Uniformly hyperbolic attractor of the Smale–Williams type for a Poincaré map in a Kuznetsov system"

D Wilczak
SIAM J.Appl.Dyn.Syst., 9(4), 12631283, 2010.

Non–autonomous coupled oscillators

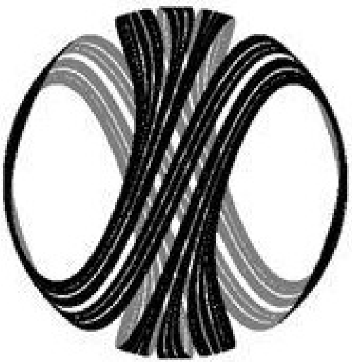
Let $a, b \in \mathbb{C}$ and consider the following system of o.d.e.'s.

$$\dot{a} = -i\epsilon(1-\sigma^2-s^2)Im(a^2\bar{b}^2)a + \frac{1}{4}i\sigma\pi(\sqrt{2}-1-2\sqrt{2}|a|^2)a$$
$$-\frac{\pi}{4}sb + \frac{1}{2}s^2(1-|a|^2-|b|^2)a$$
$$\dot{b} = i\epsilon(1-\sigma 2-s^2)Im(a^2\bar{b}^2)b + \frac{1}{4}i\sigma\pi(\sqrt{2}+1-2\sqrt{2}|b|^2)b$$
$$+\frac{\pi}{4}sa + \frac{1}{2}s^2(1-|a|^2-|b|^2)b.$$

In the sustained regime of self–oscillations the relation $|a|^2+|b|^2=1$ holds.

“Plykin type of attractor in non–autonomous coupled oscillators”

S P Kuznetsov
Chaos (19), 013114, 2009.



Exercise: Verify rigorously his statements.

The list continues.

- "Anosov parameter values for the triple linkage and a physical system with a uniformly chaotic attractor"

  T J Hunt, R S MacKay
  Nonlinearity, 16, 4, 2003.

- "Hyperbolic Plykin attractor can exist in neuron models"

  Belykh V, Belykh I, Mosekilde E
  Int.J.Bif.Ch. 15(11), 3567-3578, 2005.

- "Autonomous coupled oscillators with hyperbolic strange attractors"

  Kuznetsov S, Pikovsky A
  Physica D, 232, 87-102, 2007.

- "Robust chaos in autonomous time–delay systems"

  Arzhanukhina D S, Kuznetsov S
  to appear, 2017.

Things to remember

So:

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- It also proves that hyperbolic chaos is a stable property, thus we cannot "sweep it under the carpet".
- It teaches us that "one can not escape chaos", since chaotic sets are usually attracting. The chaotic attractors provided by the theory are also ammenable to analytical study (and really beautiful!).
- In applications of all sorts we come across hyperbolic constructions.

Why Hyperbolic Theory won't go away

- It provides the mathematical framework to discuss concepts such as "stability", "genericity", "equidistribution".
- It is present in everyday applications.
- Actually, it contains a famous branch of geometry (hyperbolic riemannian geometry).
- Dynamical systems that are not hyperbolic might be singular hyperbolic. To study Singular Hyperbolic Theory one should be familiar with... (try to guess).

For Further Reading I

Clark Robinson
Dynamical Systems
CRC Press, 1998.

Anatole Katok, Boris Hasselblatt
Introduction to the Modern Theory of Dynamical Systems
Cambridge University Press, 1995.

Stephen Smale
Differentiable dynamical systems
Bull.Am.Math.Society, 73(6):747–817, 1967.